Exercises for Quantum Mechanics
(TFFY54)

Johan Henriksson and Patrick Norman

Department of Physics, Chemistry and Biology,
Linköping University, SE-581 83 Linköping, Sweden

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For a Hermitian operator $\hat{\Omega}$, we know that
\[ \int \psi^* \hat{\Omega} \psi \, dr = \int (\hat{\Omega} \psi)^* \psi \, dr, \quad \forall \psi. \]

Show that
\[ \int \psi_1^* \hat{\Omega} \psi_2 \, dr = \int (\hat{\Omega} \psi_1)^* \psi_2 \, dr, \quad \forall \psi_1, \psi_2. \]

**Hint:** Consider a linear combination $\Psi = c_1 \psi_1 + c_2 \psi_2$, where $c_1, c_2 \in \mathbb{C}$.

2

Consider a one-dimensional harmonic oscillator with mass $m$ and characteristic frequency $\omega$. At time $t = 0$, the state is given by
\[ \Psi(x) = \frac{1}{\sqrt{2}} (\psi_0(x) + \psi_1(x)), \]
where $\psi_n(x)$ are eigenstates to the Hamiltonian with energies $E_n = \hbar \omega (n + 1/2)$.

Determine the time-dependent state vector $\Psi(x, t)$ for $t > 0$.

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Let $\psi(x)$ be a solution to the time-independent Schrödinger equation with a potential $V(x)$ that is symmetric with respect to the origin, i.e., $V(-x) = V(x)$.

a) Show that $\psi(-x)$ also is a solution with the same energy eigenvalue.

b) If the energy levels are nondegenerate (i.e., there is at most one eigenfunction associated with a given energy), show that $\psi(-x) = \psi(x)$ or $\psi(-x) = -\psi(x)$, i.e., the eigenfunctions are either symmetric or antisymmetric with respect to the origin.

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A particle (mass $m$) is incident from the left towards the potential step
\[ V(x) = \begin{cases} 0 & x \leq 0 \\ V_0 & x > 0 \end{cases} \]

The energy of the particle is $E = 2V_0$, $V_0 > 0$.

a) Solve the time-independent Schrödinger equation.

   *Note:* Since the particle is unbounded it is not possible to normalize the wave function.

b) Calculate the probability current density $j$.

c) Define and calculate the transmission $T$ using the result in b).

d) Define and calculate the reflection $R$ using the result in b).

e) Calculate $R$ and $T$ and check that $R + T = 1$. 

Show that the expectation value of the momentum operator \( \langle \hat{p} \rangle \) is real for the wave packet
\[
\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{xp/\hbar} \Phi(p) \, dp.
\]

Determine the wave function in \( x \)-space corresponding to

a) \[
\varphi(k) = \begin{cases} (2\xi)^{-1/2} & |k| \leq \xi \\ 0 & \text{otherwise} \end{cases}
\]

b) \[
\varphi(k) = \frac{1}{\sqrt{2\pi\sigma}} \exp \left( -\frac{k^2}{2\sigma^2} \right)
\]

Consider the time-dependent Schrödinger equation
\[
\imath\hbar \frac{\partial}{\partial t} \psi(r, t) = \hat{H} \psi(r, t).
\]
If the potential is time-independent, i.e., \( V(r) \neq V(r, t) \), show that it is possible to find solutions separable in space and time, i.e., \( \psi(r, t) = \psi(r) f(t) \). Find the explicit form of \( f(t) \) and show that \( \psi(r) \) is a solution of an eigenvalue problem.

A particle of mass \( m \) in a one-dimensional box
\[
V(x) = \begin{cases} 0 & 0 \leq x \leq a \\ \infty & \text{otherwise} \end{cases}
\]
is in a mixed state composed of the ground state and the first excited state. The normalized wave function can be written as
\[
\Psi(x) = c_1 \psi_1(x) + c_2 \psi_2(x),
\]
where \( c_1 \) and \( c_2 \) are constants and \( \psi_1(x) \) and \( \psi_2(x) \) are eigenfunctions corresponding to the ground state and the first excited state, respectively. The average value of the energy is \( \frac{\pi^2 \hbar^2}{ma^2} \). What can be said about \( c_1 \) and \( c_2 \)?
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If \( \langle \psi|\hat{\Omega}|\psi \rangle \) is real for all \( \psi \), show that

\[
\langle \psi_1|\hat{\Omega}|\psi_2 \rangle = \langle \psi_2|\hat{\Omega}|\psi_1 \rangle^* \]

for all \( \psi_1 \) and \( \psi_2 \). N.b., solve the problem without assuming that \( \hat{\Omega} \) is Hermitian. \textbf{Hint:} Consider the linear combinations \( \Psi = \psi_1 + \psi_2 \) and \( \Psi = \psi_1 + i\psi_2 \), respectively.

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Let \( \{\psi_n\} \) be a complete set of orthonormal functions which are solutions to the time-independent Schrödinger equation \( \hat{H}\psi_n = E_n\psi_n \). At \( t = 0 \) the system is described by the wave function

\[
\Psi(x) = \frac{1}{\sqrt{2}}e^{i\alpha}\psi_1(x) + \frac{1}{\sqrt{3}}e^{i\beta}\psi_2(x) + \frac{1}{\sqrt{6}}e^{i\gamma}\psi_3(x).
\]

a) Write down \( \Psi(x,t) \).

b) At time \( t \) a measurement of the energy of the system is performed. What is the probability to obtain the result \( E_2 \)?

c) Calculate \( \langle \hat{H} \rangle \)

d) Is the mean value of the energy equal to any of the possible outcomes of a measurement?

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A particle of mass \( m \) is moving in the one-dimensional potential

\[
V(x) = \begin{cases} 
0 & 0 \leq x \leq a \\
\infty & \text{otherwise}
\end{cases}.
\]

At a certain time the particle is in a state given by the wave function

\[
\Psi(x) = Nx(a-x)
\]

where \( N \) is a normalization constant.

a) Calculate the probability that a measurement of the energy yields the ground state energy.

b) Calculate the probability that a measurement of the energy yields a result between 0 and \( \frac{\hbar^2a^2}{2ma^2} \).
Consider a particle (mass \( m \)) in a one-dimensional box \( (0 \leq x \leq a) \). At time \( t = 0 \), the particle is described by the wave function

\[
\Psi(x) = N \left[ \sqrt{\frac{2}{a}} \sin \left( \frac{\pi}{a} x \right) + \sqrt{\frac{2}{a}} \sin \left( \frac{4\pi}{a} x \right) \right].
\]

a) Determine \( N \) and \( \Psi(x,t) \).

b) Calculate \( \langle x \rangle_t = \langle \Psi(x,t)|\hat{x}|\Psi(x,t) \rangle \).

**Hint:** \( \sin(\varphi) \cdot \sin(4\varphi) = \frac{\cos(3\varphi) - \cos(5\varphi)}{2} \)

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Verify the following relations for matrix exponentials.

a) \( \exp(\hat{A})^\dagger = \exp(\hat{A}^\dagger) \)

b) \( \hat{B} \exp(\hat{A})\hat{B}^{-1} = \exp(\hat{B}\hat{A}\hat{B}^{-1}) \)

c) \( \exp(\hat{A} + \hat{B}) = \exp(\hat{A}) \exp(\hat{B}) \) if \( [\hat{A}, \hat{B}] = 0 \)

d) \( \exp(-\hat{A}) \exp(\hat{A}) = 1 \)

e) \( \frac{d}{d\lambda} \exp(\lambda \hat{A}) = \hat{A} \exp(\lambda \hat{A}) = \exp(\lambda \hat{A}) \hat{A}, \quad \hat{A} \neq \hat{A}(\lambda) \)

f) \( \exp(-\hat{A})\hat{B} \exp(\hat{A}) = \hat{B} + [\hat{B}, \hat{A}] + \frac{1}{2!}[[\hat{B}, \hat{A}], \hat{A}] + \frac{1}{3!}[[[\hat{B}, \hat{A}], \hat{A}], \hat{A}] + \ldots \)

**Hint:** Consider the Taylor expansion of \( \exp(-\lambda \hat{A})\hat{B} \exp(\lambda \hat{A}) \) around \( \lambda = 0 \).

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Define the trace of an operator as

\[
\text{Tr}(\hat{\Omega}) = \sum_i \langle i|\hat{\Omega}|i \rangle = \sum_i \Omega_{ii}
\]

and the density operator, commonly used in many applications, as \( \hat{\rho} = |\psi\rangle\langle \psi| \).

a) Show that \( \text{Tr}(\hat{\Omega}\hat{\Lambda}) = \text{Tr}(\hat{\Lambda}\hat{\Omega}) \).

b) If the basis \( |i\rangle \) is transformed by a unitary transformation, i.e., \( |i'\rangle = \hat{U}|i\rangle \), show that the trace of the operator is unchanged in the new basis.

c) Show that \( \text{Tr}(\hat{\rho}) = 1 \).

d) Show that it is possible to use \( \hat{\rho} \) to express the expectation value of an operator as \( \langle \hat{\Omega} \rangle = \text{Tr}(\hat{\rho}\hat{\Omega}) \).

**Comment:** This means that expectation values of observables are not affected by the choice of representation (basis) we make for our wave functions since the trace is invariant under unitary transformations.
In a three-dimensional vector space, assume that we have found the commuting operators $\hat{\Omega}$ and $\hat{\Lambda}$ corresponding to some physical observables. We choose a basis $|n\rangle$, $n = \{1, 2, 3\}$, for which none of the operators are diagonal but given by the matrix representations

$$\Omega = \begin{pmatrix} 2 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 2 \end{pmatrix} \quad \text{and} \quad \Lambda = \frac{1}{2} \begin{pmatrix} 3 & -i\sqrt{2} & i \\ i\sqrt{2} & 2 & \sqrt{2} \\ -i & \sqrt{2} & 3 \end{pmatrix}.$$ 

a) Solve the eigenvalue problem $\hat{\Omega}|\omega\rangle = \omega|\omega\rangle$ to find which values of the observable $\Omega$ we can measure.

b) Since one eigenvalue is degenerated, the eigenstates are not uniquely defined through the eigenvalues $\omega$. To resolve this problem, we can use the commuting operator $\hat{\Lambda}$. Show that $\Lambda$ is block diagonal in the basis $|\omega\rangle$.

c) Diagonalize the $2 \times 2$ block in $\Lambda$ to find a basis in which both $\Omega$ and $\Lambda$ are diagonal.

d) The pairs of eigenvalues $|\omega, \lambda\rangle$ uniquely defines the eigenstates. Which are the three pairs of eigenstates?

Comment: This exercise is closely related to real problems such as the hydrogen atom where one of the observables usually is the Hamiltonian and you encounter degenerate energy levels.

Consider a Hermitian operator $\hat{\Omega}$.

a) Show that $\exp(i\hat{\Omega})$ is unitary.

b) Given the result in a), show that a wave function normalized at $t = t_0$ will remain normalized at any $t > t_0$.

c) Show that nondegenerate eigenstates of $\hat{\Omega}$ are orthogonal.

d) Show that eigenvalues and expectation values of $\hat{\Omega}$ are real.

In a three-dimensional vector space the operator $\hat{\Omega}$ can be represented as

$$\Omega = \begin{pmatrix} 2 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 2 \end{pmatrix}.$$ 

Find the matrix representation of the operator $\sqrt{\Omega}$, i.e., the operator which when squared yields the operator $\Omega$. 

Comment: This exercise is closely related to real problems such as the hydrogen atom where one of the observables usually is the Hamiltonian and you encounter degenerate energy levels.
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Let $\hat{U}(a)$ be a unitary operator defined as

$$\hat{U}(a) = e^{-ia\hat{p}/\hbar},$$

where $a$ is a real number of dimension length. Furthermore, define the transformation of an arbitrary operator $\hat{\Omega}$ as

$$\hat{\Omega} = \hat{U}^\dagger(a) \hat{\Omega} \hat{U}(a).$$

a) What does this transformation correspond to in your laboratory?

Note: The wave function will be left unchanged in this case.

b) Determine the transformed coordinate and momentum operators $\hat{x}$ and $\hat{p}$.

c) If you got the correct answers in b), it is trivial to determine the expectation values $\langle \hat{x} \rangle$ and $\langle \hat{p} \rangle$. These averages should reflect your answer in a). Determine these expectation values.

d) If we, instead of transforming the operators, transform our state vectors according to

$$|\tilde{\psi}\rangle = \hat{U}(a)|\psi\rangle,$$

what does $|\tilde{\psi}\rangle$ correspond to in your laboratory? Note that the observables, of course, will be unaltered, i.e., $\langle \tilde{\psi}|\hat{\Omega}|\tilde{\psi}\rangle = \langle \psi|\hat{\Omega}|\psi\rangle$.

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A harmonic oscillator of mass $m$ is in a state described by the wave function

$$\Psi(x,t) = \frac{1}{\sqrt{2}} e^{i\beta \psi_0(x)} e^{-iE_0 t} + \frac{1}{\sqrt{2}} e^{-i\beta \psi_1(x)} e^{-iE_1 t},$$

where $\beta$ is a real constant, $\psi_0$ and $\psi_1$ are the ground and the first excited states, respectively, and $E_0$ and $E_1$ are the corresponding energies. Determine the expectation values of $\hat{H}$ and $\hat{x}$.

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Consider a harmonic oscillator of mass $m$ with eigenstates $|\psi_n\rangle$ and energy levels $E_n = \hbar \omega (n + \frac{1}{2})$. The Hamiltonian for this system is $\hat{H} = \hat{p}^2/2m + m\omega^2 \hat{x}^2/2$. For this system, show that

$$\frac{\partial E_n}{\partial \omega} = \left\langle \psi_n \left| \frac{\partial \hat{H}}{\partial \omega} \right| \psi_n \right\rangle.$$

Comment: This is a direct result of the much more general Hellman–Feynman theorem in quantum mechanics.
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A particle of mass $m$ is located in the potential $V(x) = m\omega^2 x^2/2$. The particle is not in a stationary state and is at time $t = 0$ described by the wave function

$$|\Psi(0)\rangle = \sum_{n=0}^{\infty} c_n |n\rangle.$$ 

We can assume that the wave function is real at $t = 0$, i.e., all $c_n$ are real numbers. Show that the time-dependent average value of the position $\langle \hat{x}\rangle_t$ is

$$\langle \hat{x}\rangle_t = \langle \hat{x}\rangle_{t=0} \cos \omega t.$$ 

Comment: The average of the position is oscillating with a frequency $\omega$ (much like a classical particle). The frequency $\omega$ is related to the energy in the usual way, i.e., $E_n = \hbar \omega (n + 1/2)$.

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Consider a one-dimensional harmonic oscillator (mass $m$) in state $\psi_n$.

a) Show that the uncertainty product in this state is given by

$$(\Delta x)_n (\Delta p)_n = \left( n + \frac{1}{2} \right) \hbar, \quad n = 0, 1, 2, \ldots$$

b) What is so special with the case $n = 0$?

c) Show that

$$\langle \hat{T}\rangle_n = \langle \hat{V}\rangle_n = \frac{1}{2} E_n = \frac{1}{2} \langle \hat{H}\rangle_n.$$ 

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Consider a harmonic oscillator in the energy basis $\{|n\rangle\}$, with $\hat{H}|n\rangle = E_n |n\rangle$ and $E_n = \hbar \omega (n + 1/2)$. In order to make a transition from energy basis to a representation in ordinary space, $\psi_n(x)$, we can exploit the properties of the creation and annihilation operators, $\hat{a}^\dagger |0\rangle = |1\rangle$ and $\hat{a} |1\rangle = |0\rangle$, where

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}} \left( \xi - \frac{d}{d\xi} \right) \quad \text{and} \quad \hat{a} = \frac{1}{\sqrt{2}} \left( \xi + \frac{d}{d\xi} \right).$$

Determine $\psi_0(\xi)$ and $\psi_1(\xi)$.

24

A particle in a system with $V(x) = V(-x)$ is described by the wave function

$$\Psi(x) = \sigma^{1/2} \pi^{-1/4} e^{-x^2 \sigma^2/\Delta^2}.$$
If $\Psi(x)$ is expanded in eigenfunctions to the Hamiltonian,

$$\Psi(x) = \sum_{n=0}^{\infty} c_n \psi_n(x),$$

half of the coefficients can be determined using a simple symmetry argument. How?

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A particle is moving in a central potential, $V(r)$, corresponding to a potential function approaching zero as the distance $r$ to the center approaches infinity. The particle is in a stationary state where the time-independent part of the wave function is given by

$$\Psi(x, y, z) = N x y e^{-\alpha r},$$

where $N$ is a normalization constant and $\alpha$ is a given positive constant.

a) Calculate the possible results when measuring $\hat{L}^2$ and $\hat{L}_z$ and state the corresponding probabilities.

b) Determine the potential $V(r)$.

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A hydrogen atom is in the $2p$ state with $m_l = 0$ and $m_s = 1/2$. The system is thus represented by the wave function

$$\psi_{21,0,1/2}(r, \theta, \phi, t) = R_{21}(r) Y_{10}(\theta, \phi) (1^0_1) e^{-iE_2t/\hbar}.$$

At time $t = 0$, a measurement of the orbital angular momentum along the $x$-axis is performed.

a) Motivate why the magnitude of the orbital angular momentum remains unchanged by the measurement, i.e., the state at time $t > 0$ will still be an eigenstate of $\hat{L}^2$ with eigenvalue $l = 1$.

b) Determine the possible values and corresponding probabilities in the measurement performed at time $t = 0$.

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a) For a centrosymmetric system with $V = V(r)$, show that

$$[\hat{L}_z, \hat{V}] = 0 \quad \text{and} \quad [\hat{L}_z, \hat{p}_z^2] = 0.$$

b) Show furthermore that if $\hat{H} = \frac{\hat{p}_x^2}{2m} + \hat{V}$, it follows that

$$[\hat{L}_z, \hat{H}] = 0 \quad \text{and} \quad [\hat{L}^2, \hat{H}] = 0.$$

c) What consequences do these results have?
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For a system with $l = 1$, determine the matrix representation of $\hat{L}_z$.

29

Study a state $|\psi\rangle$ given by $\langle r, \theta, \phi | \psi \rangle = \psi(r, \theta, \phi)$ in coordinate basis. In this basis, determine the transformed state $e^{-i\hat{L}_z/h}|\psi\rangle$.

Hint: The effect of $\hat{L}_z$ in the coordinate basis is $-i\hbar \partial/\partial \phi$.

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Consider $\psi(r, \theta, \phi) = f(r) Y_{l,m}(\theta, \phi)$, where $Y_{l,m}(\theta, \phi)$ is a spherical harmonic. We want to make a simultaneous measurement of $L_x$ and $L_y$, but an uncertainty in the measurement is unavoidable.

a) Why?

b) For a given (fixed) value of the quantum number $l$, find the value of $m$ which leads to the largest possible accuracy if $L_x$ and $L_y$ are measured simultaneously.

Hint: Minimize $D(m) = (\Delta L_x)^2 + (\Delta L_y)^2$

c) What is the smallest possible accuracy if $L_x$, and $L_y$ is measured simultaneously?

d) What happens in the special case of $l = 0$?

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Consider a particle of mass $m$ moving in the potential

$$V(x) = \begin{cases} 
0 & 0 \leq x \leq a \\
\infty & \text{otherwise}
\end{cases}$$

In nonrelativistic quantum mechanics (i.e., $v \ll c$) the energy levels, as we all know, are determined by

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}, \quad n = 1, 2, 3, \ldots$$

and the corresponding normalized wave functions are given by

$$\psi_n(x) = \begin{cases} 
\frac{2}{a} \sin \left( \frac{n\pi x}{a} \right) & 0 \leq x \leq a \\
0 & \text{otherwise.}
\end{cases}$$

We want to make an approximation of the relativistic correction to the energy $E_n$ using perturbation theory. For the kinetic energy we have (according to the theory of relativity)

$$E_k = \sqrt{(pc)^2 + (mc^2)^2} - mc^2 = mc^2 \left[ \sqrt{1 + \left( \frac{p}{mc} \right)^2} - 1 \right]$$
If this is expanded for small linear momenta this yields

\[ E_k = \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \ldots, \]

which gives us a correction \(-\frac{p^4}{8m^3c^2}\) to the nonrelativistic expression \(\frac{p^2}{2m}\) for the kinetic energy.

a) What is, quantum mechanically, the perturbative part of the Hamiltonian?

b) Use this Hamiltonian to make a first-order perturbation theory calculation of the relativistic correction to the unperturbed energy level \(E_n^{(0)}\) and compare quantitatively the correction term with \(E_n^{(0)}\).

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In a hydrogen atom, the electrostatic interaction between the electron and the proton in the nucleus results in the potential energy given by (in spherical coordinates)

\[ V(r, \theta, \phi) = V(r) = -\frac{e^2}{4\pi\varepsilon_0} \frac{1}{r}. \]

The time-independent Schrödinger equation is separable, \(\Psi(r, \theta, \phi) = \psi_n(r)Y_{l,m}(\theta, \phi)\), where \(Y_{l,m}(\theta, \phi)\) are spherical harmonics. In spherical coordinates, the Hamiltonian is given by

\[ \hat{H} = -\frac{\hbar^2}{2\mu} \frac{1}{dr^2} \frac{d^2}{dr^2} + \frac{e^2}{(4\pi\varepsilon_0)r} + \hat{L}^2 \frac{I}{2I}, \]

where \(\mu\) is the reduced mass of the electron and the proton, \(I\) is the moment of inertia, and \(\hat{L}\) is the angular momentum operator.

a) Use the variational principle to estimate the ground state energy for the electron in a hydrogen atom.

**Hint:** Due to the fact that the potential energy does not depend on the angles \(\theta\) and \(\phi\), a suitable trial function for the ground state is \(\phi = \phi(r) = N \exp(-\alpha r)\), where \(\alpha \geq 0\) is the variational parameter.

b) How good/bad is the estimate? Why so?

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Consider a particle with mass \(m\) moving in the two-dimensional potential

\[ V(x, y) = \begin{cases} 0 & 0 \leq x \leq a \text{ and } 0 \leq y \leq a, \\ \infty & \text{otherwise}. \end{cases} \]

Using first-order perturbation theory, show that the degeneracy of the first excited state is lifted by the perturbation

\[ \hat{H}' = \epsilon \hat{x}^2, \]

where \(\epsilon\) is small. Visualize the energy levels schematically in a graph. Which are the proper zeroth-order eigenfunctions?
A hydrogen atom in state $|1,1,0,\pm\frac{1}{2}\rangle$ is subjected to a time-dependent magnetic field, $\mathbf{B}(r,t) = B_0 \mathbf{e}_z \sin(\omega t)$. Neglecting terms quadratic in the magnetic field, the interaction between the electron and the external magnetic field, $\mathbf{B}$, is described by the Hamiltonian

$$\hat{H}' = \frac{\mu_B}{\hbar} (\hat{\mathbf{L}} + 2\hat{\mathbf{S}}) \cdot \mathbf{B},$$

where $\mu_B$ is the Bohr magneton. Determine the possible final states. Comment on these final states in comparison with those of an perturbing electric field?

Using the variational principle, estimate the ground state energy for a one-dimensional anharmonic oscillator with potential $V(x) = \lambda x^4$. Make the assumption

$$\phi(x) = \sqrt{\frac{\beta}{\sqrt{\pi}}} e^{-\beta^2 x^2/2}, \quad \langle \phi | \phi \rangle = 1.$$

A one-dimensional harmonic oscillator (mass $m$, force $-m\omega^2 xe_x$) is perturbed by a small force $-m\epsilon x e_x$ where $0 < \epsilon \ll \omega$. Using perturbation theory, determine the corrections to the unperturbed energy levels to second order. Furthermore calculate the exact energy levels of the perturbed system and compare the results.

A particle with spin $\frac{1}{2}$ is in the state $|\alpha\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. If the spin is measured along the $e_x$-direction forming an angle $\theta$ with the $z$-axis, what are the probabilities of getting the results $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$, respectively?

Consider a particle of spin $s = 1/2$ in a time-dependent magnetic field $\mathbf{B} = B \cos(\omega t) \mathbf{e}_z$. The Hamiltonian becomes $\hat{H} = \mu_B B \hat{S}_z \cos(\omega t)$. Assume that the spin state at time $t = 0$ is $|\chi(0)\rangle = |\beta_x\rangle$.

a) Calculate $|\chi(t)\rangle$.

b) Calculate $\langle \chi(t)| \hat{S}_x |\chi(t)\rangle$. 

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In spin space it is possible to describe an infinitesimal rotation $\delta_\alpha$ around the $y$-axis by the unitary operator

$$\hat{U}(\delta_\alpha \mathbf{e}_y) = \hat{I} - \frac{i}{\hbar} \delta_\alpha \hat{S}_y.$$ 

a) Show that $\hat{U}(\alpha \mathbf{e}_y) = \exp(-i \alpha \hat{\sigma}_y/2)$.

b) Let $\hat{U}(\alpha \mathbf{e}_y)$ operate on an arbitrary state $\chi$ and show that this is equivalent to a rotation of the spin vector by an angle $\alpha/2$ in spin space.

c) What is the expected result of a rotation by the angle $2\pi$? Make a $2\pi$ rotation, and compare to the expected result. How many turns must a spin vector be rotated in order to get back to its original state?

In magnetic resonance, the magnetic moment, $\mathbf{m}$, of a particle interacts with the applied magnetic field. This interaction is described by the Hamiltonian

$$\hat{H} = -\mathbf{m} \cdot \mathbf{B},$$

where the components of the magnetic dipole moment operator are $\hat{m}_i = -\mu_B \hat{\sigma}_i$ and $\mu_B$ is the Bohr magneton. Assume that a constant magnetic field $\mathbf{B} = B\mathbf{e}_z$ is applied. At time $t = 0$, the spin is measured to be $+\hbar/2$ in the $x$-direction, i.e., $|\chi(0)\rangle = |\alpha_x\rangle$.

a) Calculate $|\chi(t)\rangle$ by using the propagator

$$|\chi(t)\rangle = e^{-i \hat{H}t/\hbar} |\chi(0)\rangle.$$ 

b) Calculate $|\chi(t)\rangle$ by solving the time-dependent Schrödinger equation using the assumption

$$|\chi(t)\rangle = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}.$$ 

c) Let $t > 0$. Calculate the probability to obtain the result $-\hbar/2$ in a measurement of the spin along the $x$-direction.

d) Let $t > 0$. Calculate the probability to obtain the result $+\hbar/2$ in a measurement of the spin along the $z$-direction.

Determine the eigenvalues of the operator $\mathbf{e}_n \cdot \hat{\mathbf{S}}$, where $\mathbf{e}_n = (\sin \theta, 0, \cos \theta)$ and $\hat{\mathbf{S}}$ is the spin operator for a particle with spin $1/2$. The eigenvalues correspond to the possible measured values when measuring spin along the direction $\mathbf{e}_n$. 

12
Consider the spin operators

\[ \hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

a) Show that \( \hat{S}_x^2 = \hat{S}_y^2 = \hat{S}_z^2 = \frac{\hbar^2}{4} \hat{I} \).

b) Determine the eigenvalues and eigenvectors to \( \hat{S}_x \) and \( \hat{S}_y \).

c) Show the anticommutation relation \([\hat{S}_x, \hat{S}_y] = \hat{S}_x \hat{S}_y + \hat{S}_y \hat{S}_x = 0\).

The spin and angular dependent part of the wave function for an electron is given by:

\[ \Phi(\theta, \phi) = \sqrt{\frac{2l}{2l+1}} Y_{l,-1}(\theta, \phi) |\alpha\rangle + \sqrt{\frac{1}{2l+1}} Y_{l,0}(\theta, \phi) |\beta\rangle \]

where \( Y_{l,m}(\theta, \phi) \) are normalized spherical harmonics (eigenfunctions to \( \hat{L}_z \) and \( \hat{L}_z \)) and

\[ |\alpha\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\beta\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

are eigenstates to \( \hat{S}_x^2 \) and \( \hat{S}_z \).

Show that \( \Phi(\theta, \phi) \) is an eigenfunction to the \( z \)-component of the angular momentum operator, \( \hat{J}_z = \hat{L}_z + \hat{S}_z \), and calculate the corresponding expectation value.

At time \( t = 0 \), the spin state of an electron is given by

\[ |\chi(0)\rangle = \frac{1}{2} \left( \begin{array}{c} \sqrt{2} \\ 1 + i \end{array} \right). \]

a) Determine the direction

\[ e_n = \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix} \]

along which to apply a magnetic field, i.e., \( \mathbf{B} = B_0 e_n \), such that, at time \( t = t_0 \), the spin state is given by

\[ |\chi(t_0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]

b) How long time will you need to apply the magnetic field?

Hint: The Hamiltonian of the system is \( \hat{H} = \mu_B \mathbf{B} \cdot \mathbf{\sigma} \), where \( \mu_B \) is the Bohr magneton.
From the point of view of the electron, the orbital motion of the proton in hydrogen creates a magnetic field of considerable strength—the numerical value of the magnetic field experienced by the electron is in the order of 0.4 Tesla. Due to this strong internal magnetic field, the energy of the atom will depend on the orientation of the electron spin.

The interaction operator that needs to be added to the nonrelativistic Hamiltonian in order to describe spin-orbit interaction is

$$\hat{H}_{SO} = \frac{\hbar \alpha}{2m^2 c r^3} \hat{\mathbf{L}} \cdot \hat{\mathbf{S}},$$

where $m$ is the electron mass, $c$ is the speed of light, and $\alpha$ is the dimensionless fine-structure constant ($\alpha \approx 1/137$).

Consider hydrogen in its $2p$-state. Without consideration made to the spin-orbit interaction this state is sixfold energy degenerate.

a) Give a complete set of commuting observables for hydrogen when spin-orbit interaction is accounted for.

b) Describe qualitatively how the energy degeneration of the six states will change in this case.

c) Determine numerical values of any energy splittings you have proclaimed above. Also, give the energy separation between the $1s$ and $2p$ levels in order to make sure that the corrections are small in absolute terms.

At time $t = t_0$, a hydrogen atom, described by the wave function

$$\psi(x, y, z, t_0) = N x exp \left( -\frac{\sqrt{x^2 + y^2 + z^2}}{2a_0} \right) \begin{pmatrix} 1 \\ -i \end{pmatrix},$$

where $N$ is a normalization constant and $a_0$ is the Bohr radius, is placed in a constant external magnetic field $\mathbf{B} = B_0 \mathbf{e}_z$. The Hamiltonian describing the interaction with the external field is $\hat{H} = \mu_B B_0 \hat{\sigma}_z$. Calculate the time-dependent expectation values of the spin projection along the $x$-, $y$-, and $z$-axes.

At time $t_0$, the wave function of a hydrogen atom is given by

$$\psi(x, y, z, t_0) = N x exp \left( -\frac{\sqrt{x^2 + y^2 + z^2}}{2a_0} \right) \begin{pmatrix} 1 \\ -i \end{pmatrix},$$

where $N$ is a normalization constant and $a_0$ is the Bohr radius.

a) If we measure the orbital angular momentum projection along the $z$-axis, which values will we measure and what are their respective probabilities?

b) If we measure the projection of the spin along the $x$-, $y$, and $z$-axes, which values will we measure and what are their respective probabilities?
48

a) Using the variational method, show that the virial theorem

\[ 2\langle \hat{T} \rangle + \langle \hat{V} \rangle = 0, \]

where \( \langle \hat{T} \rangle \) is the kinetic energy and \( \langle \hat{V} \rangle \) is the potential energy, is satisfied for a hydrogen atom.

**Hint:** Assume that the true normalized wave function is \( \Psi(x, y, z) \) and use as a normalized variational wave function

\[ \psi_\eta(x, y, z) = \eta^{3/2} \psi(\eta x, \eta y, \eta z) \]

where \( \eta \) is a scaling parameter. Here \( \eta = 1 \) corresponds to the exact solution.

b) For a hydrogen-like system in its ground state, calculate

\[ \frac{1}{\langle r \rangle}, \quad \langle \frac{1}{r} \rangle, \quad \langle \hat{p}_x^2 \rangle, \quad \langle \hat{p}_y^2 \rangle, \quad \langle \hat{p}_z^2 \rangle, \quad \text{and} \quad \langle \hat{p}^2 \rangle. \]

49

Consider a spinless particle with mass \( m \) and charge \(-e\) in the potential

\[ V(x, y) = \begin{cases} \frac{1}{2}m\omega^2 x^2, & y \in [0, a] \\ \infty, & \text{otherwise} \end{cases} \]

where

\[ \omega = \frac{3\pi^2 \hbar}{2ma^2}. \]

The system is in the first excited state when a weak electric field is applied according to \( \mathbf{E} = E\mathbf{e}_n, \ \mathbf{e}_n = \cos \theta \mathbf{e}_x + \sin \theta \mathbf{e}_y. \) To first order, determine the effect of the perturbing electric field on the energy of the system. Comment on the \( \theta \)-dependence of the perturbation.

**Hint:**

\[ \int_0^a x \sin^2 \left( \frac{n\pi x}{a} \right) dx = \frac{a^2}{4} \]

50

A hydrogen atom is in the \( 2p \) state with \( m_l = 0 \) and \( m_s = 1/2 \), and the system is thus represented by the wave function

\[ \psi_{2,1,0,1/2}(r, \theta, \phi) = R_{2,1}(r) Y_{1,0}(\theta, \phi) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]

Determine the expectation value of the total angular momentum \( \hat{J}^2 \).
A complete set of commuting observables for the hydrogen atom is given by the set of operators $\hat{H}$, $\hat{L}^2$, $\hat{L}_z$, $\hat{S}^2$, and $\hat{S}_z$ and the set common eigenkets can be denoted as $|n, l, m_l, 1/2, m_s\rangle$. The $2p$-level of the hydrogen atom is sixfold degenerate due to the different values $m_l$ and $m_s$.

Another complete set of commuting observables is given by the set of operators $\hat{H}$, $\hat{J}^2$, $\hat{J}_z$, $\hat{L}^2$, and $\hat{S}^2$ and the set common eigenkets can in this case be denoted as $|n, j, m_j, l, 1/2\rangle$. In terms of the above “old” eigenkets, derive explicit expressions for the six wave functions that correspond to the $2p$-level in this “new” basis.

The spin-orbit interaction operator is given by $\hat{H}_{SO} = A \hat{L} \cdot \hat{S}$, where $A$ is a scalar that is independent of $(\theta, \phi)$. By re-writing the spin-orbit operator, motivate why this problems involving this operator should be solved in the basis $|j, m_j\rangle$ that are eigenvectors of $\hat{J}^2$ and $\hat{J}_z$.

Assume that the electron in hydrogen is in a $d$-orbital, i.e., $l = 2$. If one considers the interaction between the spin and orbital motion, this level will no longer be 10-fold energy degenerate. The energy operator that describes the interaction is $\hat{H}_{SO} = A \hat{L} \cdot \hat{S}$, where $A$ is a scalar that is independent of $(\theta, \phi)$.

a) The possible values of $j$ are $l \pm 1/2$, what are the interaction energies in these two cases?

b) Express $|j, m_j\rangle = |5/2, 5/2\rangle$ in terms of the “old” basis vectors $|l, m_l, s, m_s\rangle$. Show with an explicit calculation that your linear combination of old basis vectors is an eigenstate of $\hat{J}^2$ and $\hat{J}_z$.

Let us consider a hydrogen atom described by the wave function

$$\psi(r) = \frac{1}{\sqrt{5}} R_{2.1}(r) \begin{bmatrix} \sqrt{2} Y_{1,0}(\theta, \phi) \\ Y_{1,1}(\theta, \phi) \end{bmatrix}.$$ 

a) Is this state an eigenstate to total angular momentum operators $\hat{J}^2$ and $\hat{J}_z$, and, if so, what are the values of the corresponding quantum numbers $j$ and $m_j$?
b) With use of first-order perturbation theory, determine the shift in energy relative to the 2p-level in hydrogen as due to spin-orbit coupling.

**Hint:** The spin-orbit coupling operator is \( \hat{H}_{SO} = \frac{\hbar}{2m} \frac{1}{r} \mathbf{L} \cdot \mathbf{S} \), where \( m \) is the electron mass, \( c \) is the speed of light, and \( \alpha \) is the dimensionless fine-structure constant (\( \alpha \approx 1/137 \)). The following integral may be needed: \( \langle r^{-3} \rangle_{R,1} = a_0^{-3}/24 \), where \( a_0 \) is the Bohr radius.

At \( t = 0 \) the wave function for the electron in hydrogen is given by

\[
\psi(x, y, z, 0) = N y z e^{-r/3a_0} \left( \begin{array}{c} 1 \\ 2i \end{array} \right),
\]

where \( N \) is a normalization constant. Determine the possible values and probabilities in measurements of \( \hat{J}_z \), \( \hat{L}_z \) and \( \hat{S}_z \).
Answers

2. \( \Psi(x, t) = \frac{1}{\sqrt{2}}(\psi_0(x)e^{-i\omega t/2} + \psi_1(x)e^{-i3\omega t/2}) \)

4.

a) \( \Psi(x, t) = \begin{cases} A_r e^{i(2\sqrt{mV_0}x/\hbar - \omega t)} + A_r e^{-i(2\sqrt{mV_0}x/\hbar + \omega t)}, & x \leq 0 \left( A_r = \frac{\sqrt{2} - 1}{\sqrt{2} + 1} A_i \right) \\ A_t e^{i(\sqrt{2mV_0}x/\hbar - \omega t)}, & x > 0 \left( A_t = \frac{2\sqrt{2}}{\sqrt{2} + 1} A_i \right) \end{cases} \)

b) \( j_x = \begin{cases} 2\sqrt{\frac{V_0}{m}}(|A_i|^2 - |A_r|^2) = \frac{8\sqrt{2}}{3+2\sqrt{2}} \sqrt{\frac{V_0}{m}} |A_i|^2, & x \leq 0 \\ \frac{2\sqrt{2}}{3+2\sqrt{2}} \sqrt{\frac{V_0}{m}} |A_i|^2, & x > 0 \end{cases} \)

c) \( T = \frac{4\sqrt{2}}{3+2\sqrt{2}} \)

d) \( R = \frac{3-2\sqrt{2}}{3+2\sqrt{2}} \)

6.

a) \( \psi(x, 0) = (\pi \xi)^{-1/2} \sin(\xi x)/x \)

b) \( \psi(x, 0) = (2\pi)^{-1/2} \exp(-\sigma^2 x^2/2) \)

7.

a) \( N = 2^{-1/2} \)

\( \Psi(x, t) = a^{-1/2} \left( \sin(\pi x/a)e^{-i\pi^2 h t/2m a^2} + \sin(4\pi x/a)e^{-i8\pi^2 h t/ma^2} \right) \)

8.

\[ |c_1|^2 = \frac{2}{3} \]

\[ |c_2|^2 = \frac{1}{3} \]

10.

a) \( \Psi(x, t) = \frac{1}{\sqrt{2}} e^{i(\alpha - E_1 t/\hbar)} \psi_1 + \frac{1}{\sqrt{3}} e^{i(\beta - E_2 t/\hbar)} \psi_2 + \frac{1}{\sqrt{6}} e^{i(\gamma - E_3 t/\hbar)} \psi_3 \)

b) \( \mathcal{P}_2 = 1/3 \)

c) \( \langle \hat{H} \rangle = \frac{E_1}{3} + \frac{E_2}{3} + \frac{E_3}{6} \)

d) No

11.

\[ \mathcal{P}(E_1) = |c_1|^2 = \frac{960}{\pi^6} \]

\[ \mathcal{P}(E_2) = |c_2|^2 = 0 \]

\[ \mathcal{P}(0 \leq E \leq \frac{9\pi^2 a^2}{m\omega}) = |c_1|^2 + |c_2|^2 = \frac{960}{\pi^6} \]
12.  
   a) \( N = 2^{-1/2} \)  
   \[ \Psi(x, t) = a^{-1/2} \left( \sin \left( \frac{\pi x}{a} \right) \exp \left( -i \frac{\pi^2 \hbar t}{2ma^2} \right) + \sin \left( \frac{4\pi x}{a} \right) \exp \left( -i \frac{4\pi^2 \hbar t}{2ma^2} \right) \right) \]  
   b) \( \langle \hat{x} \rangle_t = \frac{a}{2} - \frac{2a}{\pi^2} \left[ \frac{1}{3} - \frac{1}{2} \right] \cos \left( \frac{E_4 - E_1}{\hbar^2} t \right) \)

15.  
   a) \( \omega_1 = 1 \leftarrow |\omega_1 \rangle = 2^{-1/2}(1, 0, i)^T \)  
   \( \omega_2 = 1 \leftarrow |\omega_2 \rangle = (0, 1, 0)^T \)  
   \( \omega_3 = 3 \leftarrow |\omega_3 \rangle = 2^{-1/2}(1, 0, -i)^T \)  
   b) \( U^\dagger \Omega U = \begin{pmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \)  
   c) \( |\omega_1, \lambda_1 \rangle = 2^{-1/2}(|\omega_1 \rangle - i|\omega_2 \rangle) \)  
   \( |\omega_2, \lambda_2 \rangle = 2^{-1/2}(|\omega_1 \rangle + i|\omega_2 \rangle) \)  
   \( |\omega_3, \lambda_3 \rangle = |\omega_3 \rangle \)  
   d) \( |\omega, \lambda \rangle = \{|1, 0\rangle, |1, 2\rangle, |3, 2\rangle\} \)

17.  
   \[ \sqrt{\Omega} = \frac{1}{2} \begin{pmatrix} 1 + \sqrt{3} & 0 & (\sqrt{3} - 1)i \\ 0 & 2 & 0 \\ (1 - \sqrt{3})i & 0 & 1 + \sqrt{3} \end{pmatrix} \]

18.  
   a) The system is fixed, but you are performing measurement w.r.t. to a translated coordinate system.  
   b) \( \tilde{x} = \hat{x} + a \) and \( \tilde{p} = \hat{p} \)  
   c) \( \langle \tilde{x} \rangle = \langle \hat{x} \rangle + a \) and \( \langle \tilde{p} \rangle = \langle \hat{p} \rangle \)  
   d) The system is moved within the laboratory.

19.  
   \( \langle E \rangle = \frac{1}{2}(E_0 + E_1) = \hbar \omega \) where \( \omega = \frac{E_0 + E_1}{2\hbar} \).  
   \( \langle \hat{x} \rangle = \left[ \frac{\hbar}{2\pi m \omega} \right]^\dagger \cos(\omega t + 2\beta) \)

23.  
   \( \psi_0(\xi) = \frac{i}{\sqrt{\pi \sigma}} e^{-\xi^2/2} \)  
   \( \psi_1(\xi) = \frac{2^{1/2}}{\sqrt{\pi \sigma}} \xi e^{-\xi^2/2} \)

24.  
   \( c_n = 0 \) if \( n \) is an odd number.
25.
   a) The only possible measurement result for $\hat{L}^2$ is $6\hbar^2$ (with probability 1).
   Possible results when measuring $\hat{L}_z$ are $\pm 2\hbar$, each with probability $\frac{1}{2}$.
   b) The potential is $V(r) = -\frac{3\hbar^2}{mr}$.

26.
   a) $\hat{L}^2$ and $\hat{L}_z$ are commuting observables.
   b) The measured values are $\pm \hbar$, both with probability $\frac{1}{2}$.

28.
   
   
   $L^2 = \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

29.
   
   
   $e^{-i\phi_0 \hat{L}_z / \hbar} |\psi\rangle = |\psi(r, \theta, \phi - \phi_0)\rangle$

30.
   a) $[\hat{L}_x, \hat{L}_y] \neq 0$
   b) $D(m) = l(l+1)\hbar^2 - m^2\hbar^2$
   c) $\hbar^2(l+1)$
   d) $D(m) = 0$, i.e., it is possible to determine both $\hat{L}_x$ and $\hat{L}_y$ simultaneously.

31.
   b) $E_{\text{pert}}^n = E_0^n \left(1 - \frac{e^a}{2m^2}\right)$

32.
   
   
   a) $E_0 \leq -\frac{ae^2}{3ze^{-\phi_0}}$

33.
   
   The shifts in energy are $\left[\frac{1}{3} - \frac{1}{2\pi^2}\right]ea^2$ and $\left[\frac{1}{3} - \frac{1}{8\pi^2}\right]ea^2$, respectively.

34.
   
   $\Delta l = 0$, \quad \begin{cases} 
   \Delta m_l = \pm 1, & \Delta m_l = 0 \\
   \Delta m_l = 0, & \Delta m_l = -1 
   \end{cases}$

35.
   
   $E_{\text{min}} = \frac{\lambda^{1/3}}{4} \left(\frac{\hbar^2}{2m}\right)^{2/3} \lambda^{1/3}$

36.
   
   $E_{\text{pert}}^n = E_0^n = \hbar\omega \left(n + \frac{1}{2}\right) - \frac{e^2}{2m^2\omega}$
37. \[
\mathcal{P}_\alpha = \cos^2 \frac{\theta}{2} \\
\mathcal{P}_\beta = \sin^2 \frac{\theta}{2}
\]

38. 
\[|\chi(t)\rangle = \left[ \exp \left( -i \mu_B B \frac{t}{\hbar} \sin(\omega t) \right) |\alpha_z\rangle - \exp \left( i \mu_B B \frac{t}{\hbar} \sin(\omega t) \right) |\beta_z\rangle \right] / \sqrt{2} \]

b) \[\langle \chi(t) | \hat{S}_x | \chi(t) \rangle = -\frac{\hbar}{2} \cos \left( \frac{2 \mu_B B}{\hbar} \sin(\omega t) \right) \]

40. 
\[|\chi(t)\rangle = [\exp(-i \mu_B B t / \hbar) |\alpha_z\rangle + \exp(+i \mu_B B t / \hbar) |\beta_z\rangle] / \sqrt{2} \]

c) \[\mathcal{P}_{\alpha_x} = \cos^2 \left( \frac{\mu_B B t}{\hbar} \right) \]
d) \[\mathcal{P}_{\alpha_x} = \frac{1}{2} \]

41. \[\lambda = \pm \frac{\hbar^2}{2} \]

42. 
\[\hat{S}_x: \frac{\hbar}{2} \leftrightarrow \frac{1}{\sqrt{2}} (\begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} 1 \\ -i \end{pmatrix}) \]
\[\hat{S}_y: \frac{\hbar}{2} \leftrightarrow \frac{1}{\sqrt{2}} (\begin{pmatrix} i \\ 1 \end{pmatrix}, -\begin{pmatrix} 1 \\ i \end{pmatrix}) \]

43. \[\langle \hat{J}_z \rangle = (l - \frac{1}{2}) \hbar \]

44. 
\[a) \epsilon_n = 2^{-1/2}(1, -1, 0)^T \]
\[b) \tau_0 = \frac{\pi \hbar}{4 \mu_B B} \]

45. 
\[a) \hat{H}, \hat{J}^2, \hat{J}_z, \hat{L}^2, \text{ and } \hat{S}^2 \]
\[c) \Delta E \approx 4.5 \cdot 10^{-5} \text{ eV} \]

46. 
\[\langle \hat{S}_x \rangle_t = \frac{\hbar}{2} \sin \left( \frac{\mu_B B}{\hbar} (t - \tau_0) \right) \]
\[\langle \hat{S}_y \rangle_t = -\frac{\hbar}{2} \cos \left( \frac{\mu_B B}{\hbar} (t - \tau_0) \right) \]
\[\langle \hat{S}_z \rangle_t = 0 \]
47. 

a) Measuring $L_z$ yields $\pm\hbar$, both with probability 1/2.

b) Measuring $S_x$ yields $\pm\hbar/2$, both with probability 1/2.

Measuring $S_y$ yields $-\hbar/2$.

Measuring $S_z$ yields $\pm\hbar/2$, both with probability 1/2.

48. 

b) $\frac{1}{(\ell)} = \frac{2Z_{\ell,0}}{\ell}$

$\langle \downarrow \rangle = \frac{Z_{\ell,0}}{\ell}$

$\langle \hat{p}_x^2 \rangle = \langle \hat{p}_y^2 \rangle = \langle \hat{p}_z^2 \rangle = \frac{1}{3} \left( \frac{\hbar Z_{\ell,0}}{\ell} \right)^2$

49. First excited state is double degenerate with $E_1 = E_2 = \frac{11\pi^2 \hbar^2}{4ma^2}$.

The first-order energy correction is the same for both states and equals $\Delta E_1 = \Delta E_2 = \frac{1}{2}aeE \sin \theta$.

50. $\langle \hat{J}^2 \rangle = \frac{1}{2} \hbar^2$

51. The $|j, m_j, l, 1/2\rangle$ kets take the form

$$Y_{l, m_j}^{j=1/2, \pm 1/2}(\theta, \phi) = \pm \sqrt{\frac{\ell + m_j + 1/2}{\ell + 1}} Y_{l, m_j - 1/2}(\theta, \phi) \quad Y_{l, m_j + 1/2}(\theta, \phi)$$

53. a) $\frac{A\hbar^2}{2} \times \left\{ \begin{array}{ll} l; & j = l + 1/2 \\
                               -l - 1; & j = l - 1/2 \end{array} \right.$

54.

a) $j = 3/2, m_j = 1/2$

b) $\langle \hat{H}_{SO} \rangle = \frac{\hbar^2 \alpha}{\sqrt{m_e c a_0}}$

55. The measurements of angular momenta give:

<table>
<thead>
<tr>
<th>Operator</th>
<th>Value</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_z$</td>
<td>$\frac{\hbar}{2}$</td>
<td>1/5</td>
</tr>
<tr>
<td></td>
<td>$-\frac{\hbar}{2}$</td>
<td>4/5</td>
</tr>
<tr>
<td>$\hat{L}_z$</td>
<td>$\hbar$</td>
<td>1/2</td>
</tr>
<tr>
<td></td>
<td>$-\hbar$</td>
<td>1/2</td>
</tr>
<tr>
<td>$\hat{J}_z$</td>
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</tr>
<tr>
<td></td>
<td>$\hbar/2$</td>
<td>4/10</td>
</tr>
<tr>
<td></td>
<td>$-\hbar/2$</td>
<td>1/10</td>
</tr>
<tr>
<td></td>
<td>$-3\hbar/2$</td>
<td>4/10</td>
</tr>
</tbody>
</table>
Summary

Commutator relations

\[ [\hat{A}, \hat{B} \hat{C}] = [\hat{A}, \hat{B}] \hat{C} + \hat{B} [\hat{A}, \hat{C}] \]
\[ e^{\hat{A}\hat{B}e^{-\hat{A}}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \frac{1}{3!} [\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \ldots \]

Time evolution

\[ |\psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\psi(0)\rangle \quad \frac{d}{dt} (\Omega) = \frac{1}{i\hbar} \left\langle [\hat{\Omega}, \hat{A}] \right\rangle + \left\langle \frac{\partial \hat{\Omega}}{\partial t} \right\rangle \]

Uncertainty relations

\[ \frac{1}{2} |\langle [\hat{\Omega}, \hat{\Lambda}] \rangle| \leq \Delta \Omega \Delta \Lambda \]

Momentum basis

\[ \langle x|p \rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ipx}{\hbar}\right) \]

Translations and rotations

\[ \hat{U}_T(\alpha e_n) = \exp\left(\frac{-i\alpha}{\hbar} e_n \cdot \hat{p}\right) \quad \hat{U}_R(\alpha e_n) = \exp\left(\frac{-i\alpha}{\hbar} e_n \cdot \hat{J}\right) \]

Harmonic oscillator

\[ \hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger) \quad \hat{p} = i\sqrt{\frac{m\hbar\omega}{2}} (\hat{a}^\dagger - \hat{a}) \]
\[ \hat{a}|n\rangle = \sqrt{n}|n-1\rangle \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \]

Angular momentum

\[ \hat{J} \times \hat{J} = i\hbar \hat{J} \]
\[ \hat{J}_x = \hat{J}_x \pm i\hat{J}_y \quad \hat{J}_y = \frac{1}{2}(\hat{J}_+ + \hat{J}_-) \quad \hat{J}_z = \frac{1}{2}(\hat{J}_- - \hat{J}_+) \]
\[ \hat{J}_+|j, m\rangle = \hbar \sqrt{j(j+1) - m(m+1)}|j, m+1\rangle = \hbar \sqrt{j(j+1)(j+m+1)}|j, m+1\rangle \]

Spin angular momentum

\[ |\alpha_n\rangle = \left(\begin{array}{c} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{-i\phi/2} \end{array}\right) \quad |\beta_n\rangle = \left(\begin{array}{c} -\sin \frac{\theta}{2} e^{-i\phi/2} \\ \cos \frac{\theta}{2} e^{i\phi/2} \end{array}\right) \]

Variational principle

\[ E_0 \leq \frac{\langle \Psi|\hat{H}|\Psi\rangle}{\langle \Psi|\Psi\rangle} \]

Perturbation theory

**Time-independent, nondegenerate:**

\[ E_n^{(1)} = \langle \psi_n|\hat{H}'|\psi_n\rangle \quad |\phi_n^{(1)}\rangle = -\sum_{m \neq n} \langle \psi_m|\hat{H}'|\psi_n\rangle |\psi_m\rangle \quad E_n^{(2)} = \langle \psi_n|\hat{H}'|\phi_n^{(1)}\rangle \]

**Time-independent, degenerate:**

\[ \det \left\{ \mathbf{H}_n' - E_n^{(1)} \mathbf{I} \right\} = 0 \]

**Time-dependent:**

\[ d_n^{(1)}(t) = \frac{1}{i\hbar} \int_0^t dt' e^{i\omega_n t'} H_n'(t') \quad P_{f_1}(t) = \frac{1}{\hbar^2} \int_0^t dt' e^{i\omega_{f_1} t'} H_{f_1}'(t') \]

\[ \hat{H}'(t) = \hat{A}_1 \sin(\omega t) \quad P_{f_1}(t, \omega) = \frac{|A_{f_1}|^2}{4\hbar^2} \left| \frac{1 - e^{i(\omega_{f_1} + \omega)t}}{2\omega} - \frac{1 - e^{i(\omega_{f_1} - \omega)t}}{2\omega} \right|^2 \]