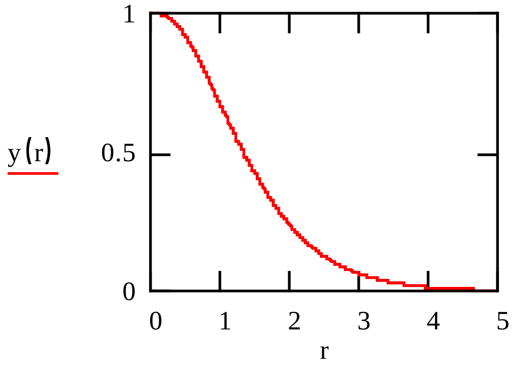
1 $\mathbf{E}=\hat{r}\frac{Q(r)}{r^2}$, where Q(r) is all charge within the distance r from the origin.

$$Q(r)=e+\int_0^r dr4\pi r^2\left[-\frac{e}{\pi a_0^3}\exp(-2r/a_0)\right]=\left|x=2r/a_0\right|=e-\frac{4\pi e}{\pi a_0^2}\left(\frac{a_0}{2}\right)^3\int_0^{2r/a_0}dxx^2\left[\exp(-x)\right]$$

$$=e-ea_0\left[1-\exp(\frac{-2r}{a_0})\left(2\left(\frac{r}{a_0}\right)^2+2\left(\frac{r}{a_0}\right)+1\right)\right]=e\exp(\frac{-2r}{a_0})\left(2\left(\frac{r}{a_0}\right)^2+2\left(\frac{r}{a_0}\right)+1\right)$$

$$\underline{\mathbf{E}(\mathbf{r})=\hat{r}\frac{e}{r^2}\exp(\frac{-2r}{a_0})\left(2\left(\frac{r}{a_0}\right)^2+2\left(\frac{r}{a_0}\right)+1\right)}$$

$$y(r):=\exp(-2\cdot r)\cdot\left(2\cdot r^2+2\cdot r+1\right)$$



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2 Do we have a monopole contribution? Yes we have a net charge: $\Phi^{(1)}=q/r$.

Do we have a dipole contribution? No because of the symmetry: $\Phi^{(2)}=0$.

Quadrupole moment: For symmetry reasons Q11 and Q22 are equal and we only need to calculate the third diagonal element. Eq (2.4.3) is then also valid even though we don't have axial symmetry.

$$Q=\int_V\rho(\mathbf{r}')\left(3z'^2-r'^2\right)dx'dy'dz'=\int_S\rho_s(\mathbf{r}')\left(3z'^2-r'^2\right)dx'dy'$$

$$=\frac{q}{a^2}\int_{-a/2}^{a/2}dx'\int_{-a/2}^{a/2}dy'\left(-r'^2\right)=-\frac{q}{a^2}\int_{-a/2}^{a/2}dx'\int_{-a/2}^{a/2}dy'\left(x'^2+y'^2\right)$$

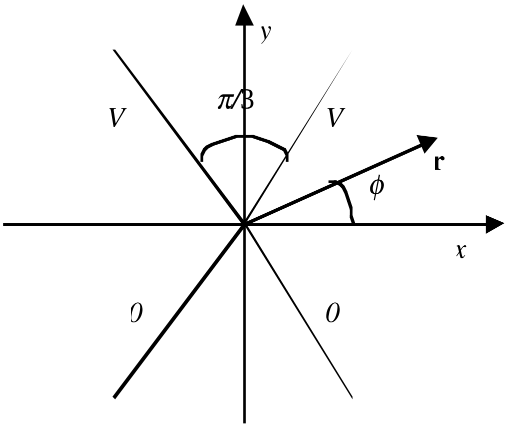
$$=-\frac{q}{a^2}\int_{-a/2}^{a/2}dx'\left[y'x'^2+\frac{y'^3}{3}\right]_{y'=-a/2}^{a/2}=-\frac{q}{a^2}\int_{-a/2}^{a/2}dx'\left[ax'^2+\frac{a^3}{12}\right]$$

$$=-\frac{q}{a^2}\left[\frac{ax'^3}{3}+\frac{a^3x'}{12}\right]_{-a/2}^{a/2}=-\frac{q}{a^2}\left[\frac{a^4}{12}+\frac{a^4}{12}\right]=-\frac{qa^2}{6}$$

$$\Phi^{(4)}=\frac{Q}{4}\frac{1}{r^3}\left(3\cos^2\theta-1\right)=-\frac{qa^2}{24}\frac{1}{r^3}\left(3\cos^2\theta-1\right)$$

The total potential is $\underline{\Phi=\frac{q}{r}-\frac{qa^2}{24}\frac{1}{r^3}\left(3\cos^2\theta-1\right)\ldots}$

3.



This is a two-dimensional problem. The plates divide the whole 3D space into four regions: upper, lower, right and left. The upper region is bounded by two plates at the constant potential V. $\Phi(r,\phi)\equiv V$ is a solution in this region and since these boundary conditions leads to a unique solution this is the solution. In the lower region the same type of argument gives that the solution is $\Phi(r,\phi)\equiv 0$. In the right region the imaginary part of the analytic function ln(z) is a proper candidate for the solution or rather

$$\Phi(r,\phi)=V/2+V\,\mathrm{Im}\left[\ln(x+iy)\right]/(2\pi/3)=V/2+(3V/2\pi)\arg\left[\ln(x+iy)\right]$$
$$=V/2+(3V/2\pi)\phi;\ -\pi/3\le\phi\le\pi/3$$

From the symmetry of the problem follows that one gets the result in the left region from the above result by replacing ϕ with $\pi-\phi$, i.e., $\Phi(r,\phi)=2V-3\phi V/2\pi;\ 2\pi/3\le\phi\le 4\pi/3$.
The equi-potential surfaces are planes through the z-axis.
The E-field is obtained from $\mathbf{E}=-\nabla\Phi=-\frac{\partial\Phi}{\partial r}\hat{r}-\frac{1}{r}\frac{\partial\Phi}{\partial\phi}\hat{\phi}-\frac{\partial\Phi}{\partial z}\hat{z}=-\frac{1}{r}\frac{\partial\Phi}{\partial\phi}\hat{\phi}$. We use cylinder coordinates.
In the upper and lower regions the field is zero. In the right region it is $\mathbf{E}=-(3V/2\pi r)\hat{\phi}$ and in the left it is $\mathbf{E}=(3V/2\pi r)\hat{\phi}$.
The surface charge density on the upper V-shaped plate is the jump in the normal component of the electric field when going from inside the metal to outside divided by 4π: $\rho(r)=|\mathbf{E}|/4\pi=(3V/8\pi^2 r)$. For the lower plate the electric field points into the plates and $\rho(r)=-|\mathbf{E}|/4\pi=-(3V/8\pi^2 r)$. We note that the surface charge densities diverge at the edges, which is what one expects to find.

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4 In order for the fields to vanish inside the sphere the total (free plus bound) charge density on the surface layer of the sphere has to be constant, the same in the two regions. Due to the different dielectric properties of the media the bound surface charge densities is different in the two regions of the spherical surface; so are the free surface charge densities.

$\rho_{f,1}+\rho_{b,1}=\rho_{f,2}+\rho_{b,2}$

The total free charge density adds up to Q.

$$\rho_{f,1}+\rho_{f,2}=2\rho_f=Q/\left(4\pi a^2\right)$$

To find the bound charges we use Gauss' theorem on a Gauss pillbox at the surface of the sphere. The bottom surface should be inside the metal and the upper inside the dielectric. Remember that only the free charges contribute.

$$D_1=4\pi\rho_{f,1};\;\; D_2=4\pi\rho_{f,2}$$

We repeat with the corresponding equation for the **E**-field.

$$D_1/\varepsilon_1=E_1=4\pi\left(\rho_{f,1}+\rho_{b,1}\right);\;\; D_2/\varepsilon_2=E_2=4\pi\left(\rho_{f,2}+\rho_{b,2}\right)$$

$$\rho_{f,1}+\rho_{b,1}=\rho_{f,1}/\varepsilon_1;\;\; \rho_{f,2}+\rho_{b,2}=\rho_{f,2}/\varepsilon_2$$

$$\rho_{f,1}/\varepsilon_1=\rho_{f,2}/\varepsilon_2;\;\; \rho_{f,2}=2\rho_f-\rho_{f,1}$$

$$\downarrow$$

$$\rho_{f,1}\varepsilon_2=\rho_{f,2}\varepsilon_1=2\rho_f\varepsilon_1-\rho_{f,1}\varepsilon_1$$

$$\downarrow$$

$$\rho_{f,1}=\frac{2\rho_f\varepsilon_1}{\varepsilon_1+\varepsilon_2}\to\rho_{f,2}=\frac{2\rho_f\varepsilon_2}{\varepsilon_1+\varepsilon_2}$$

$$\rho_{b,1}=\rho_{f,1}\left(\frac{1}{\varepsilon_1}-1\right)=\frac{2\rho_f\left(1-\varepsilon_1\right)}{\varepsilon_1+\varepsilon_2}$$

$$\rho_{b,2}=\rho_{f,2}\left(\frac{1}{\varepsilon_2}-1\right)=\frac{2\rho_f\left(1-\varepsilon_2\right)}{\varepsilon_1+\varepsilon_2}$$

From this we get

$$\rho_{f,1}+\rho_{b,1}=\frac{2\rho_f}{\varepsilon_1+\varepsilon_2};\;\; \rho_{f,2}+\rho_{b,2}=\frac{2\rho_f}{\varepsilon_1+\varepsilon_2}$$

and finally

$$\mathbf{E}=\begin{cases}\dfrac{2}{\varepsilon_1+\varepsilon_2}\dfrac{Q}{r^2}\hat{r},\; r>a\\ 0,\; r<a\end{cases}$$

$$\mathbf{D}=\begin{cases}\dfrac{2\varepsilon_1}{\varepsilon_1+\varepsilon_2}\dfrac{Q}{r^2}\hat{r},\ \text{in the liquid}\\ \dfrac{2\varepsilon_2}{\varepsilon_1+\varepsilon_2}\dfrac{Q}{r^2}\hat{r},\ \text{in the gas}\\ 0,\ \text{in the sphere}\end{cases}$$