

- 1) $\Phi(r) = q_0 \frac{e^{-r/a}}{r} \rightarrow \mathbf{E} = q_0 \frac{e^{-r/a}}{r^2} \left(1 + \frac{r}{a}\right) \hat{r}$; Gauss' law gives that the total charge, $Q(r)$ within a sphere of radius r is $Q(r) = \frac{4\pi r^2}{4\pi} E(r) = r^2 E(r) = q_0 e^{-r/a} \left(1 + \frac{r}{a}\right)$. This charge includes the original charge at the origin. We may here see that this quantity goes to zero as the radius goes to infinity, which answers the second question. Now, $\frac{dQ(r)}{dr} = -\frac{q_0 r}{a^2} e^{-r/a}$ and

$$q_{ind} = \int_0^\infty dr \frac{dQ(r)}{dr} = \int_0^\infty 4\pi r^2 dr \frac{1}{4\pi r^2} \frac{dQ(r)}{dr} = \int d^3r \frac{1}{4\pi r^2} \frac{dQ(r)}{dr} = \int d^3r \rho_{ind}(\mathbf{r}); \text{ thus}$$

a) $\rho_{ind}(\mathbf{r}) = \frac{1}{4\pi r^2} \frac{dQ(r)}{dr} = -\frac{q_0}{4\pi r a^2} e^{-r/a} = -\frac{1}{4\pi a^2} \Phi(r)$

- b) We have already found that $q_{ind} = -q_0$ but let us also rederive this with direct integration:

$$\begin{aligned} q_{ind} &= \int d^3r \rho_{ind}(\mathbf{r}) = \int_0^\infty dr \frac{dQ(r)}{dr} = -\frac{q_0}{a^2} \int_0^\infty dr r e^{-r/a} = /r \rightarrow ra/ = -q_0 \int_0^\infty dr r e^{-r} \\ &= -q_0 \left[-r e^{-r} \Big|_0^\infty + \int_0^\infty dr e^{-r} \right] = -q_0 \left[-r e^{-r} \Big|_0^\infty - e^{-r} \Big|_0^\infty \right] = -q_0 [0 - 0 - 0 + 1] = -q_0 \end{aligned}$$

- 2a) Monopole moment: 0 (total charge is zero)

Dipole moment: 0 (inversion symmetry)

$$Q_{ij} = \int_{-\infty}^{+\infty} dx_1 \int_{-\infty}^{+\infty} dx_2 \int_{-\infty}^{+\infty} dx_3 \rho(\mathbf{r}) (3x_i x_j - r^2 \delta_{ij})$$

$$\begin{aligned} Q_{11} &= \int_{-a}^a dx \rho_0 (3x^2 - x^2) + \int_{-a}^a dy \rho_0 (3 \cdot 0^2 - y^2) + \int_0^{2\pi} d\varphi a \rho_1 (3(a \cos \varphi)^2 - a^2) \\ &= \rho_0 \left(4 \frac{a^3}{3} - 2 \frac{a^3}{3} \right) + \rho_1 2\pi a \left(3 \frac{a^2}{2} - a^2 \right) = \frac{2a^2 q}{3} - 2a^2 q = -\frac{4a^2 q}{3} \end{aligned}$$

$$\begin{aligned} Q_{22} &= \int_{-a}^a dy \rho_0 (3y^2 - y^2) + \int_{-a}^a dx \rho_0 (3 \cdot 0^2 - x^2) + \int_0^{2\pi} d\varphi a \rho_1 (3(a \sin \varphi)^2 - a^2) \\ &= \rho_0 \left(4 \frac{a^3}{3} - 2 \frac{a^3}{3} \right) + \rho_1 2\pi a \left(3 \frac{a^2}{2} - a^2 \right) = \frac{2a^2 q}{3} - 2a^2 q = -\frac{4a^2 q}{3} \end{aligned}$$

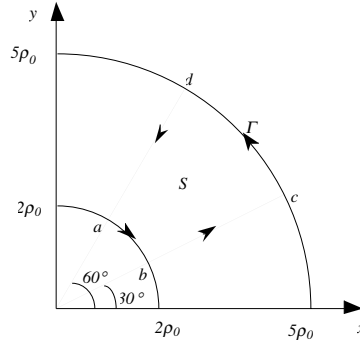
$$\begin{aligned} Q = Q_{33} &= \int_{-a}^a dx \rho_0 (3 \cdot 0^2 - x^2) + \int_{-a}^a dy \rho_0 (3 \cdot 0^2 - y^2) + \int_0^{2\pi} d\varphi a \rho_1 (3 \cdot 0^2 - a^2) \\ &= \rho_0 \left(-2 \frac{a^3}{3} - 2 \frac{a^3}{3} \right) + \rho_1 2\pi a (-a^2) = -\frac{4a^2 q}{3} + 4a^2 q = \frac{8a^2 q}{3} = -Q_{11} - Q_{22} \end{aligned}$$

$$Q_{12} = Q_{21} = Q_{13} = Q_{31} = Q_{23} = Q_{32} = 0$$

2b)

$$\begin{aligned}\Phi(r, \theta) &= \Phi^{(1)}(r, \theta) + \Phi^{(2)}(r, \theta) + \Phi^{(4)}(r, \theta) + \dots \\ &= 0 + 0 + \frac{1}{4} Q \frac{(3 \cos^2 \theta - 1)}{r^3} + \dots = \frac{2a^2 q}{3} \frac{(3 \cos^2 \theta - 1)}{r^3} + \dots\end{aligned}$$

3.



$$(a) \quad \oint_{\Gamma} \mathbf{E} \cdot d\mathbf{l} = \int_a^b \mathbf{E} \cdot d\mathbf{l} + \int_b^c \mathbf{E} \cdot d\mathbf{l} + \int_c^d \mathbf{E} \cdot d\mathbf{l} + \int_d^a \mathbf{E} \cdot d\mathbf{l}$$

$$\begin{aligned}\int_a^b \mathbf{E} \cdot d\mathbf{l} &= \int_{\varphi=60^\circ}^{\varphi=30^\circ} E_0 \left[(2\rho_0/\rho_0) \cos \varphi \hat{\rho} + \sin \varphi \hat{\phi} \right] \cdot 2\rho_0 d\varphi \hat{\phi} = \int_{\varphi=60^\circ}^{\varphi=30^\circ} E_0 [\sin \varphi] \cdot 2\rho_0 d\varphi = E_0 2\rho_0 [-\cos \varphi]_{\varphi=60^\circ}^{\varphi=30^\circ} \\ &= -E_0 \rho_0 (\sqrt{3} - 1)\end{aligned}$$

$$\begin{aligned}\int_b^c \mathbf{E} \cdot d\mathbf{l} &= \int_{\rho=2\rho_0}^{\rho=5\rho_0} E_0 \left[(\rho/\rho_0) \cos 30^\circ \hat{\rho} + \sin 30^\circ \hat{\phi} \right] \cdot d\rho \hat{\rho} = \int_{\rho=2\rho_0}^{\rho=5\rho_0} E_0 (\rho/\rho_0) \cos 30^\circ \cdot d\rho = E_0 (1/\rho_0) \cos 30^\circ \left[\rho^2/2 \right]_{\rho=2\rho_0}^{\rho=5\rho_0} \\ &= E_0 \rho_0 21\sqrt{3}/4\end{aligned}$$

$$\begin{aligned}\int_c^d \mathbf{E} \cdot d\mathbf{l} &= \int_{\varphi=30^\circ}^{\varphi=60^\circ} E_0 \left[(5\rho_0/\rho_0) \cos \varphi \hat{\rho} + \sin \varphi \hat{\phi} \right] \cdot 5\rho_0 d\varphi \hat{\phi} = \int_{\varphi=30^\circ}^{\varphi=60^\circ} E_0 [\sin \varphi] \cdot 5\rho_0 d\varphi = E_0 5\rho_0 [-\cos \varphi]_{\varphi=30^\circ}^{\varphi=60^\circ} \\ &= E_0 \rho_0 5(\sqrt{3} - 1)/2\end{aligned}$$

$$\begin{aligned}\int_d^a \mathbf{E} \cdot d\mathbf{l} &= \int_{\rho=5\rho_0}^{\rho=2\rho_0} E_0 \left[(\rho/\rho_0) \cos 60^\circ \hat{\rho} + \sin 60^\circ \hat{\phi} \right] \cdot d\rho \hat{\rho} = \int_{\rho=5\rho_0}^{\rho=2\rho_0} E_0 (\rho/\rho_0) \cos 60^\circ \cdot d\rho = E_0 (1/\rho_0) \cos 60^\circ \left[\rho^2/2 \right]_{\rho=5\rho_0}^{\rho=2\rho_0} \\ &= -E_0 \rho_0 21/4\end{aligned}$$

$$\oint_{\Gamma} \mathbf{E} \cdot d\mathbf{l} = E_0 \rho_0 \left[(-\sqrt{3} + 1) + 21\sqrt{3}/4 + 5(\sqrt{3} - 1)/2 - 21/4 \right] = E_0 \rho_0 \frac{27}{4} (\sqrt{3} - 1)$$

$$(b) \quad \nabla \times \mathbf{E} = \hat{\rho}[0] + \hat{\phi}[0] + \hat{z} \left[\frac{\partial}{\partial \rho} (\rho \sin \varphi) - \frac{\partial}{\partial \varphi} ((\rho/\rho_0) \cos \varphi) \right] = \hat{z} \frac{E_0 \sin \varphi}{\rho} [1 + \rho/\rho_0]$$

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = \int_S \left(\frac{E_0 \sin \varphi}{\rho} [1 + \rho/\rho_0] \hat{z} \right) \cdot (\rho d\rho d\varphi \hat{z}) = \int_{\varphi=30^\circ}^{\varphi=60^\circ} \int_{\rho=2\rho_0}^{\rho=5\rho_0} d\rho d\varphi E_0 \sin \varphi [1 + \rho/\rho_0]$$

$$= E_0 \int_{\varphi=30^\circ}^{\varphi=60^\circ} d\varphi \sin \varphi \int_{\rho=2\rho_0}^{\rho=5\rho_0} d\rho [1 + \rho/\rho_0] = E_0 [-\cos \varphi]_{\varphi=30^\circ}^{\varphi=60^\circ} \left[\rho + \rho^2/2\rho_0 \right]_{\rho=2\rho_0}^{\rho=5\rho_0} = E_0 \rho_0 \frac{27}{4} (\sqrt{3} - 1)$$

which agrees with the result in (a). Q.E.D.

4.

a)

$$\text{EMF} = \oint_c \mathbf{E} \cdot d\mathbf{l} = \int_0^a dx \hat{x} \cdot \underbrace{[A \sin(kx - \omega t)]}_{0} \hat{y} + \int_0^b dy \hat{y} \cdot [A \sin(ka - \omega t)] \hat{y} + \int_a^0 dx \hat{x} \cdot \underbrace{[A \sin(kx - \omega t)]}_{0} \hat{y} \\ + \int_b^0 dy \hat{y} \cdot [A \sin(k \cdot 0 - \omega t)] \hat{y} = \int_0^b dy [A \sin(ka - \omega t) - A \sin(-\omega t)] = \underline{Ab [\sin(ka - \omega t) + \sin(\omega t)]}$$

b)

$$\text{EMF} = \frac{1}{c} \int_s -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = |\mathbf{B} \parallel \hat{z}| = \frac{1}{c} \int_0^a dx \int_0^b dy [-A(-\omega) \cos(kx - \omega t)] = \frac{A\omega b}{c} \int_0^a dx \cos(kx - \omega t) \\ = \frac{A\omega b}{c} \left[\frac{\sin(kx - \omega t)}{k} \right]_{x=0}^a = \frac{A\omega b}{ck} [\sin(ka - \omega t) - \sin(-\omega t)] = \underline{Ab [\sin(ka - \omega t) + \sin(\omega t)]}$$

c)

$$\text{EMF} = Ab [\sin(ka - \omega t) + \sin(\omega t)] = Ab 2 \sin\left(\frac{ka - \omega t + \omega t}{2}\right) \cos\left(\frac{ka - \omega t - \omega t}{2}\right) \\ = Ab 2 \underbrace{\sin\left(\frac{ka}{2}\right)}_{\text{Constant}} \underbrace{\cos\left(\frac{ka}{2} - \omega t\right)}_{\text{Oscillating with amplitude 1}}$$

The signal disappears completely for $ka/2 = n\pi$, $n = 0, 1, 2, \dots$, i.e. for

$$(2\pi/\lambda)a/2 = n\pi \rightarrow \underline{a = n\lambda; n = 0, 1, 2, \dots}$$

Largest signal for $ka/2 = \pi/2 + n\pi$, $n = 0, 1, 2, \dots$

$$(2\pi/\lambda)a/2 = \pi/2 + n\pi \rightarrow \underline{a = \lambda/2 + n\lambda, n = 0, 1, 2, \dots}$$
