1. Monopole contribution: Total charge zero. $\Phi^{(1)} = 0$.
Dipole contribution: $\Phi^{(2)} = 2qa\cos(\theta)/r^2$
Quadrupole contribution: Axial symmetry. $Q = q\left(3a^2 - a^2\right) - q\left(3a^2 - a^2\right) = 0$. $\Phi^{(4)} = 0$.
Octupole contribution:

$$\Phi^{(8)} = \frac{(-1)^3}{3!}\left[qa^3\frac{\partial^3}{\partial z^3}\left(\frac{1}{r}\right) + (-q)(-a)^3\frac{\partial^3}{\partial z^3}\left(\frac{1}{r}\right)\right] = -\frac{1}{3}qa^3\frac{\partial^3}{\partial z^3}\left(\frac{1}{r}\right) = -\frac{1}{3}qa^3\left(\frac{9z}{r^5} - \frac{15z^3}{r^7}\right)$$

$$= -qa^3\left[3\cos(\theta) - 5\cos^3(\theta)\right]/r^4$$

So we have the potential is $\underline{\Phi = 2qa\cos(\theta)/r^2 - qa^3\left[3\cos(\theta) - 5\cos^3(\theta)\right]/r^4 + \cdots}$.

An <u>alternative derivation</u> is to use the expansion we used at the lecture on multipole expansions

$$\frac{1}{R_{1,2}} = \left(r^2 + a^2 \mp 2ra\cos\theta\right)^{-1/2} = \frac{1}{r}\left(1 + \left(\frac{a}{r}\right)^2 \mp 2\left(\frac{a}{r}\right)\cos\theta\right)^{-1/2}$$

$$= \frac{1}{r}\left[1 \pm \left(\frac{a}{r}\right)\cos\theta + \frac{1}{2}\left(\frac{a}{r}\right)^2\left(3\cos^2\theta - 1\right) \pm \frac{1}{2}\left(\frac{a}{r}\right)^3\left(5\cos^3\theta - 3\cos\theta\right) + \ldots\right]$$

$$\Phi(r,\theta,\varphi) = q\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = q2\frac{1}{r}\left[\left(\frac{a}{r}\right)\cos\theta + \frac{1}{2}\left(\frac{a}{r}\right)^3\left(5\cos^3\theta - 3\cos\theta\right) + \ldots\right]$$

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2. $\dfrac{V}{\pi/3}\arg(z)$ is the imaginary part of an analytic function, $\dfrac{V}{\pi/3}\ln(z)$, and hence is a solution to Laplace's equation. It furthermore satisfies the boundary conditions and is therefore the unique solution we are looking for.
a) $\underline{\Phi(x,y) = 3V\arctan(y/x)/\pi}$
b) Gauss' law gives:

$$\sigma(x,0) = -\frac{1}{4\pi}\left.\frac{\partial\Phi(x,y)}{\partial y}\right|_{y=0} = -\frac{3V}{4\pi^2}\left.\frac{\arctan(y/x)}{\partial y}\right|_{y=0} = -\frac{3V}{4\pi^2}\left.\frac{1/x}{1+(y/x)^2}\right|_{y=0} = \underline{-\frac{3V}{4\pi^2 x}}$$

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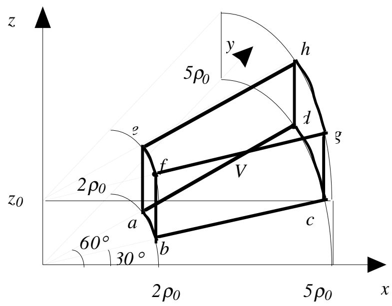
3.




(a) $\oint_S \mathbf{E}\cdot d\mathbf{S} = \int\limits_{adcb}\mathbf{E}\cdot d\mathbf{S} + \int\limits_{efgh}\mathbf{E}\cdot d\mathbf{S} + \int\limits_{abfe}\mathbf{E}\cdot d\mathbf{S} + \int\limits_{cdhg}\mathbf{E}\cdot d\mathbf{S} + \int\limits_{aehd}\mathbf{E}\cdot d\mathbf{S} + \int\limits_{bcgf}\mathbf{E}\cdot d\mathbf{S}$. The first two give no contribution since the field is perpendicular to the surface normal. $\int\limits_{adcb}\mathbf{E}\cdot d\mathbf{S} = 0$; $\int\limits_{efgh}\mathbf{E}\cdot d\mathbf{S} = 0$;

$$\int_{abfe} \mathbf{E}\cdot d\mathbf{S} = \int_{\varphi=30^\circ}^{\varphi=60^\circ}\int_{z=0}^{z=z_0}\left(E_0\left[\left(2\rho_0/\rho_0\right)\cos\varphi\hat{\rho}+\sin\varphi\hat{\varphi}\right]\right)\cdot\left(2\rho_0 d\varphi dz\left(-\hat{\rho}\right)\right) = -E_0 4\rho_0 \int_{\varphi=30^\circ}^{\varphi=60^\circ} d\varphi\cos\varphi\int_{z=0}^{z=z_0} dz$$

$$= -E_0 4\rho_0\left[\sin\varphi\right]_{\varphi=30^\circ}^{\varphi=60^\circ}\left[z\right]_0^{z_0} = -E_0\rho_0 z_0 2\left(\sqrt{3}-1\right)$$

$$\int_{cdhg} \mathbf{E}\cdot d\mathbf{S} = \int_{\varphi=30^\circ}^{\varphi=60^\circ}\int_{z=0}^{z=z_0}\left(E_0\left[\left(5\rho_0/\rho_0\right)\cos\varphi\hat{\rho}+\sin\varphi\hat{\varphi}\right]\right)\cdot\left(5\rho_0 d\varphi dz\hat{\rho}\right) = E_0 25\rho_0 \int_{\varphi=30^\circ}^{\varphi=60^\circ} d\varphi\cos\varphi\int_{z=0}^{z=z_0} dz$$

$$= E_0 25\rho_0\left[\sin\varphi\right]_{\varphi=30^\circ}^{\varphi=60^\circ}\left[z\right]_0^{z_0} = E_0\rho_0 z_0\, 25\left(\sqrt{3}-1\right)/2$$

$$\int_{aehd} \mathbf{E}\cdot d\mathbf{S} = \int_{\rho=2\rho_0}^{5\rho_0}\int_{z=0}^{z_0}\left(E_0\left[\left(\rho/\rho_0\right)\cos 60^\circ\hat{\rho}+\sin 60^\circ\hat{\varphi}\right]\right)\cdot\left(d\rho dz\hat{\varphi}\right) == E_0\sin 60^\circ\int_{\rho=2\rho_0}^{5\rho_0} d\rho\int_{z=0}^{z_0} dz =$$

$$E_0\frac{\sqrt{3}}{2}3\rho_0 z_0 = E_0\rho_0 z_0\frac{3\sqrt{3}}{2}$$

$$\int_{bcgf} \mathbf{E}\cdot d\mathbf{S} = \int_{\rho=2\rho_0}^{5\rho_0}\int_{z=0}^{z_0}\left(E_0\left[\left(\rho/\rho_0\right)\cos 30^\circ\hat{\rho}+\sin 30^\circ\hat{\varphi}\right]\right)\cdot\left(d\rho dz\left(-\hat{\varphi}\right)\right) = -E_0\sin 30^\circ\int_{\rho=2\rho_0}^{5\rho_0} d\rho\int_{z=0}^{z_0} dz$$

$$= -E_0\frac{1}{2}3\rho_0 z_0 = -E_0\rho_0 z_0\frac{3}{2}$$

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = 0+0+E_0\rho_0 z_0\left[-2\left(\sqrt{3}-1\right)+25\left(\sqrt{3}-1\right)/2+\frac{3\sqrt{3}}{2}-\frac{3}{2}\right]$$

$$= E_0\rho_0 z_0\frac{\sqrt{3}\left(-4+25+3\right)+4-25-3}{2} = \underline{E_0\rho_0 z_0 12\left(\sqrt{3}-1\right)}$$

(b) $$\nabla\cdot\mathbf{E} = \frac{E_0}{\rho}\frac{\partial}{\partial\rho}\left[\rho\left(\rho/\rho_0\right)\cos\varphi\right]+\frac{E_0}{\rho}\frac{\partial}{\partial\varphi}\left(\sin\varphi\right)+0 = E_0\left[\left(2/\rho_0\right)\cos\varphi+\left(1/\rho\right)\cos\varphi\right] = \frac{E_0\cos\varphi}{\rho}\left[1+2\rho/\rho_0\right]$$

$$\int_V\left(\nabla\cdot\mathbf{E}\right)dV = \int_{\rho=2\rho_0}^{5\rho_0}\int_{\varphi=30^\circ}^{60^\circ}\int_{z=0}^{z_0}\frac{E_0\cos\varphi}{\rho}\left[1+2\rho/\rho_0\right]\left(\rho d\rho d\varphi dz\right) = E_0\int_{\rho=2\rho_0}^{5\rho_0} d\rho\left[1+2\rho/\rho_0\right]d\varphi\int_{\varphi=30^\circ}^{60^\circ}\cos\varphi\int_{z=0}^{z_0} dz$$

$$= E_0\left[\rho+\rho^2/\rho_0\right]_{\rho=2\rho_0}^{5\rho_0}\left[\sin\varphi\right]_{\varphi=30^\circ}^{60^\circ}\left[z\right]_0^{z_0} = \underline{E_0\rho_0 z_0 12\left(\sqrt{3}-1\right)}$$

which agrees with the result in (b). Q.E.D.

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4. The force acting on the charge $q$ towards the center of the orbit must be $F = mv^2/R$ to keep the charge in orbit. It is $F = qvB(R)/c$. This means that $v = qRB(R)/mc$ and $\dot{v} = qR\dot{B}(R)/mc$ must hold. If we now change the **B** field within the orbit there will be a net magnetic flux change through the surface defined by the orbit: $\dot{\Phi} = \pi R^2\dot{\bar{B}}(R)$. This gives rise to an EMF and the electric field along the circumference of the circle is $2\pi RE = \dot{\Phi}/c \to E = \dot{\Phi}/(2\pi Rc)$. This means a force that accelerates the charge

$F_\theta = qE = q\dot{\Phi}/(2\pi Rc) = q\pi R^2\dot{\bar{B}}(R)/(2\pi Rc) = qR\dot{\bar{B}}(R)/(2c)$. So $\dot{v} = F_\theta/m = qR\dot{\bar{B}}(R)/(2mc)$.

Equating this with the result we had before gives

$$\dot{v} = qR\dot{B}(R)/mc = qR\dot{\bar{B}}(R)/(2mc) \to \dot{\bar{B}}(R) = 2\dot{B}(R) \quad Q.E.D.$$