1) The charge distribution has an axis of symmetry (the z-axis) which means:

$$\tilde{\mathbf{Q}} = \begin{pmatrix} -Q/2 & 0 & 0 \\ 0 & -Q/2 & 0 \\ 0 & 0 & Q \end{pmatrix}$$; We only need to determine one element!

$$Q = \int_V \rho(\mathbf{r}')\left(3x_3'^2 - r'^2\right)dx_1'dx_2'dx_3' = \int_V bz^4\left(3z^2 - r^2\right)dxdydz$$

$$= \int 2\pi\sin\theta d\theta r^2 drb(r\cos\theta)^4\left[3(r\cos\theta)^2 - r^2\right] = 2\pi b\int_{-1}^{1}dx\int_0^a drr^2r^4x^4\left[3r^2x^2 - r^2\right]$$

$$= 2\pi b\int_{-1}^{1}dxx^4\left[3x^2 - 1\right]\int_0^a drr^8 = 2\pi b\frac{a^9}{9}\left[\frac{3x^7}{7} - \frac{x^5}{5}\right]_{-1}^{1} = 2\pi b\frac{a^9}{9}\frac{16}{35}$$

We want to express this in terms of the total charge of the sphere.

$$q = \int d^3r\rho(\mathbf{r}) = 2\pi\int_{-1}^{1}dx\int_0^a drr^2b(rx)^4 = 2\pi b\left[\frac{r^7}{7}\right]_0^a\left[\frac{x^5}{5}\right]_{-1}^{1} = 2\pi b\frac{a^7}{7}\frac{2}{5} = 2\pi b\frac{2a^7}{35}$$

$$Q = \underline{\frac{8qa^2}{9}}$$

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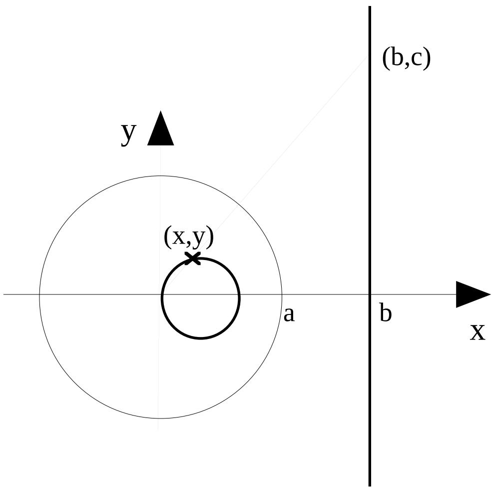
2)



We have: $\sqrt{c^2 + b^2}\sqrt{x^2 + y^2} = a^2$ from the rule of position of mirror charge and

$\frac{c}{b} = \frac{y}{x}$ from the geometry

$$\rightarrow \sqrt{y^2b^2/x^2 + b^2}\sqrt{x^2 + y^2} = a^2 \rightarrow x^2 + y^2 = \frac{a^2x}{b} \rightarrow \underline{\left(x - a^2/2b\right)^2 + y^2 = \left(a^2/2b\right)^2}$$

Thus the locus is a circle of radius $a^2/2b$ with its center at ($a^2/2b$,0).

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3)

$$\mathbf{E}'\cdot\mathbf{B}' = E_1'B_1' + E_2'B_2' + E_3'B_3' = \gamma\left(E_1-\beta B_2\right)\gamma\left(B_1+\beta E_2\right) + \gamma\left(E_2+\beta B_1\right)\gamma\left(B_2-\beta E_1\right) + E_3B_3$$
$$= \gamma^2\left[E_1B_1\left(1-\beta^2\right) + E_2B_2\left(1-\beta^2\right)\right] + E_3B_3 = E_1B_1 + E_2B_2 + E_3B_3 = \mathbf{E}\cdot\mathbf{B} \quad Q.E.D.$$

or

$$\mathbf{E}'\cdot\mathbf{B}' = \mathbf{E}'_{//}\cdot\mathbf{B}'_{//} + \mathbf{E}'_{\perp}\cdot\mathbf{B}'_{\perp} = \mathbf{E}_{//}\cdot\mathbf{B}_{//} + \gamma\left(\mathbf{E}_{\perp} + \frac{1}{c}\mathbf{v}\times\mathbf{B}_{\perp}\right)\cdot\gamma\left(\mathbf{B}_{\perp} - \frac{1}{c}\mathbf{v}\times\mathbf{E}_{\perp}\right)$$
$$= \mathbf{E}_{//}\cdot\mathbf{B}_{//} + \gamma^2\mathbf{E}_{\perp}\cdot\mathbf{B}_{\perp} - \gamma^2\left(\frac{1}{c}\mathbf{v}\times\mathbf{B}_{\perp}\right)\cdot\left(\frac{1}{c}\mathbf{v}\times\mathbf{E}_{\perp}\right) + \underbrace{\gamma^2\mathbf{B}_{\perp}\cdot\left(\frac{1}{c}\mathbf{v}\times\mathbf{B}_{\perp}\right)}_{0} - \underbrace{\gamma^2\mathbf{E}_{\perp}\cdot\left(\frac{1}{c}\mathbf{v}\times\mathbf{E}_{\perp}\right)}_{0}$$
$$= \mathbf{E}_{//}\cdot\mathbf{B}_{//} + \gamma^2\mathbf{E}_{\perp}\cdot\mathbf{B}_{\perp} - \frac{\gamma^2}{c^2}\mathbf{v}\cdot\left(\mathbf{E}_{\perp}\times\left(\mathbf{v}\times\mathbf{B}_{\perp}\right)\right) = \mathbf{E}_{//}\cdot\mathbf{B}_{//} + \gamma^2\mathbf{E}_{\perp}\cdot\mathbf{B}_{\perp} - \frac{\gamma^2}{c^2}\mathbf{v}\cdot\left(\mathbf{v}\left(\mathbf{E}_{\perp}\cdot\mathbf{B}_{\perp}\right) - \mathbf{B}_{\perp}\underbrace{\left(\mathbf{E}_{\perp}\cdot\mathbf{v}\right)}_{0}\right)$$
$$= \mathbf{E}_{//}\cdot\mathbf{B}_{//} + \gamma^2\mathbf{E}_{\perp}\cdot\mathbf{B}_{\perp} - \gamma^2\beta^2\mathbf{E}_{\perp}\cdot\mathbf{B}_{\perp} = \mathbf{E}_{//}\cdot\mathbf{B}_{//} + \mathbf{E}_{\perp}\cdot\mathbf{B}_{\perp} = \mathbf{E}\cdot\mathbf{B} \quad Q.E.D.$$

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4) a) Let the charge on the outer cylinder be Q and on the inner $-Q$. Use Gauss' law on integral form on a coaxial cylindrical surface wit radius in between 0.25 cm and 1 cm to find the **D** and **E**-fields between the metallic surfaces. With the fields in the radial direction we have $\int DdA = -4\pi Q \rightarrow \int \varepsilon EdA = -4\pi Q \rightarrow \varepsilon E 2\pi rL = -4\pi Q \rightarrow E = -2Q/\left(\varepsilon rL\right)$. This means that the potential difference is

$$V = -\int_a^b drE = \int_a^b dr\, 2Q/\left(\varepsilon rL\right) = \frac{2Q}{\varepsilon L}\ln\left(b/a\right) \rightarrow \frac{C}{L} = \frac{Q}{VL} = \frac{\varepsilon}{2\ln\left(b/a\right)} = \underline{\frac{10}{2\ln\left(4\right)} = \frac{5}{\ln\left(4\right)}}$$

b) $\omega = \left(z-a\right)/\left(az-1\right)$. Let, e.g., 0.5 be mapped onto $-d$ and 0 be mapped onto d. This gives

$$\left.\begin{aligned} -d &= \frac{0.5-a}{a0.5-1} \rightarrow -d = \frac{1-2a}{a-2} \\ d &= \frac{-a}{-1} = a \end{aligned}\right\} \rightarrow -a = \frac{1-2a}{a-2} \rightarrow -a^2 + 2a = 1-2a \rightarrow a^2 - 4a + 1 = 0$$

$\rightarrow a = 2 \pm \sqrt{3}$. There are two possibilities. The plus sign leads to an outer circle larger than the unit circle and the unit circle is the inner circle. The minus sign has the effect that the outer circle is the unit circle and the inner circle is smaller than the unit circle. The plus sign gives

$$\frac{C}{L} = \frac{\varepsilon}{2\ln\left(\frac{2+\sqrt{3}}{1}\right)} = \underline{\frac{5}{\ln\left(2+\sqrt{3}\right)}} \text{ and the minus sign } \frac{C}{L} = \frac{\varepsilon}{2\ln\left(\frac{1}{2-\sqrt{3}}\right)} = \underline{\frac{5}{\ln\left(2+\sqrt{3}\right)}}$$

which is the same result, as it should be. Numerically the value goes from 3.6 to 3.8 so the effect of the mistake was not that dramatic.
