1) For symmetry reasons the field will be $\mathbf{E} = E(r)\hat{\mathbf{r}}$. Using Gauss' law on integral form:

$$4\pi r^2 E(r) = 4\pi\rho_{encl} = 4\pi\int_0^r dr 4\pi r^2\rho(r) = (4\pi)^2\int_0^r drr^2\rho_a\cdot(r/a)e^{-(r/a)} = (4\pi)^2 a^3\rho_a\int_0^{r/a} drr^3e^{-r}$$

$$= (4\pi)^2 a^3\rho_a\left[6 - e^{-(r/a)}\left(6 + 6(r/a) + 3(r/a)^2 + (r/a)^3\right)\right]$$

$$\boxed{E(r) = \frac{q}{r^2}\left\{1 - e^{-(r/a)}\left[1 + (r/a) + \tfrac{1}{2}(r/a)^2 + \tfrac{1}{6}(r/a)^3\right]\right\}}$$

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2) $$\mathbf{S}_a = \frac{c}{4\pi}(\mathbf{E}_a\times\mathbf{B}_a) \propto \underbrace{(\mathbf{R}\times(\mathbf{R}\times\mathbf{a}))}_{\mathbf{R}(\mathbf{R}\cdot\mathbf{a})-\mathbf{a}(\mathbf{R}\cdot\mathbf{R})}\times\left(\mathbf{R}\times\underbrace{(\mathbf{R}\times(\mathbf{R}\times\mathbf{a}))}_{\mathbf{R}(\mathbf{R}\cdot\mathbf{a})-\mathbf{a}(\mathbf{R}\cdot\mathbf{R})}\right) = \left(\mathbf{R}(\mathbf{R}\cdot\mathbf{a}) - \mathbf{a}R^2\right)\times\left(0 - \mathbf{R}\times\mathbf{a}R^2\right)$$

$$= -R^2\left[(\mathbf{R}\cdot\mathbf{a})\underbrace{(\mathbf{R}\times(\mathbf{R}\times\mathbf{a}))}_{\mathbf{R}(\mathbf{R}\cdot\mathbf{a})-\mathbf{a}(\mathbf{R}\cdot\mathbf{R})} - R^2\underbrace{(\mathbf{a}\times(\mathbf{R}\times\mathbf{a}))}_{\mathbf{R}(\mathbf{a}\cdot\mathbf{a})-\mathbf{a}(\mathbf{a}\cdot\mathbf{R})}\right] = -R^2\left[\mathbf{R}\left((\mathbf{R}\cdot\mathbf{a})^2 - R^2a^2\right) + \mathbf{a}\left(R^2(\mathbf{R}\cdot\mathbf{a}) - R^2(\mathbf{R}\cdot\mathbf{a})\right)\right]$$

$$= R^2\mathbf{R}\left(R^2a^2 - (\mathbf{R}\cdot\mathbf{a})^2\right)$$

and $|\mathbf{R}\times\mathbf{a}|^2 = (\mathbf{R}\times\mathbf{a})\cdot(\mathbf{R}\times\mathbf{a}) = \mathbf{R}\cdot\underbrace{(\mathbf{a}\times(\mathbf{R}\times\mathbf{a}))}_{\mathbf{R}(\mathbf{a}\cdot\mathbf{a})-\mathbf{a}(\mathbf{a}\cdot\mathbf{R})} = R^2a^2 - (\mathbf{R}\cdot\mathbf{a})^2$

Thus, $\mathbf{S}_a \propto R^2\mathbf{R}\left(R^2a^2 - (\mathbf{R}\cdot\mathbf{a})^2\right) = R^2|\mathbf{R}\times\mathbf{a}|^2\mathbf{R} \propto |\mathbf{R}\times\mathbf{a}|^2$. Q.E.D.

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3) Do we have any monopole contribution? No there is no net charge. Do we have any dipole contribution? No, we have inversion symmetry with respect to the origin which means no dipole moment. Do we have enough symmetry around the *z*-axis for eqn. (2.43) to be valid? Yes we have but it is not so obvious! To be on the safe side we use eqns. (2.34) and the generalization of (2.35) to the one-dimensional charge distributions we have in the problem, in order to find the quadrupole contribution. If we number the rings from 1 to 3, starting with the one that cuts through the *x*-axes and move counter-clock-wise we get the table of data:

α	q	x	y	z	r
1	q	$a\sin(\theta')$	0	$a\cos(\theta')$	a
2	q	$a\sin(\theta')/2$	$a\sqrt{3}\sin(\theta')/2$	$a\cos(\theta')$	a
3	q	$-a\sin(\theta')/2$	$a\sqrt{3}\sin(\theta')/2$	$a\cos(\theta')$	a

; $$Q_{ij} = \sum_\alpha \frac{q_\alpha}{2\pi}\int_{-\pi}^{\pi} d\theta'\left(3x'_{\alpha,i}\,x'_{\alpha,j} - a^2\delta_{ij}\right)$$

We have let the angle run from -π to π for each ring and it is zero in the positive *z*-direction.

Using this equation we have

$$Q_{11} = q\left[\left(3a^2/2 - a^2\right) + \left(3a^2/8 - a^2\right) + \left(3a^2/8 - a^2\right)\right] = -3qa^2/4$$

$$Q_{12} = q\left[0 + 3\sqrt{3}a^2/8 - 3\sqrt{3}a^2/8\right] = 0$$

$$Q_{13} = q[0 + 0 + 0] = 0$$

$$Q_{21} = Q_{12} = 0$$

$$Q_{22} = q\left[\left(-a^2\right) + \left(9a^2/8 - a^2\right) + \left(9a^2/8 - a^2\right)\right] = -3qa^2/4$$

$$Q_{23} = q[0 + 0 + 0] = 0$$

$$Q_{31} = Q_{13} = 0$$

$$Q_{32} = Q_{23} = 0$$

$$Q_{33} = q\left[\left(3a^2/2 - a^2\right) + \left(3a^2/2 - a^2\right) + \left(3a^2/2 - a^2\right)\right] = 3qa^2/2 = Q$$

$$\Phi^{(4)}(r,\theta,\varphi)=\frac{1}{6r^5}\left[-\frac{3qa^2}{4}\left(3x^2-r^2\right)-\frac{3qa^2}{4}\left(3y^2-r^2\right)+\frac{3qa^2}{2}\left(3z^2-r^2\right)\right]$$

$$=\frac{3qa^2}{8}\frac{\left(3\cos^2\theta-1\right)}{r^3}=\frac{Q}{4}\frac{\left(3\cos^2\theta-1\right)}{r^3}$$

Answer: $\Phi(r,\theta,\varphi)=\Phi^{(4)}(r,\theta,\varphi)=\underline{\frac{3qa^2}{8}\frac{\left(3\cos^2\theta-1\right)}{r^3}}$

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4) The function $\Phi(x,y)=\Phi(\theta)=V\theta/2\pi$ obeys the boundary conditions and since it is the imaginary part of the analytic function $F(z)=V\log(z)/2\pi$, $z=x+iy=r\exp(i\theta)$, $0<\theta<2\pi$, it is our solution. The electric field is $\mathbf{E}=-\nabla\Phi(\theta)=-\mathbf{e}_\theta\frac{1}{r}d\Phi/d\theta=-(V/2\pi r)\mathbf{e}_\theta$. The field lines form circles around the origin and the equipotential curves are rays through the origin. These results are good if we are not interested in the details very near the edge of the capacitor. We should be at distances from the edge that are large compared to the separation between the plates but still small compared to the size of the capacitor. So if we go back and study our results from the lecture we find that those asymptotically approach the present results when we go away from the capacitor edge.

Thus the answer is: The potential is $\Phi(\theta)=\underline{V\theta/2\pi}$ and the electric field is $\mathbf{E}=\underline{-(V/2\pi r)\mathbf{e}_\theta}$.