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CONFORMAL MAPPING

In a limited group of problems one can use a short cut to the solution of the Laplace's equation, conformal mapping. The problem has to be 2D (two dimensional), i.e., the boundaries, boundary conditions and sources contain only two variables, x and y say.

Some examples where this method can be used is: the electrostatic potential; heat conduction; flow of fluids.

We will of course focus on the electrostatic potential. The idea is that the problem at hand with a certain geometry may be mapped into a problem with simpler geometry or with a geometry that we have already solved.

For polygonal boundaries a general method of constructing the mapping is offered by *Schwartz-Cristoffel* transformations. For non-polygonal boundaries one must rely on "dictionaries" of mappings. One such dictionary with 30 different mappings is found in a student text book by R. V. Churchill[1]. Another dictionary is found in a book by K. J. Binns and P. J. Lawrenson[2]. For further reading about conformal mapping see W. R. Smythe[3].

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[1] R. V. Churchill: *Complex variables and applications*, 2nd Edition, McGraw-Hill, New York 1960.

[2] K. J. Binns and P. J. Lawrenson, *Analysis and Computation of Electric and Magnetic Field Problems,* 2nd Edition, Pergamon Press, New York 1973.

[3] W. R. Smythe, *Static and Dynamic Electricity*, 3rd Edition, Taylor&Frances 1989.

The conformal mapping relies on the properties of *analytic functions*. Thus we need to refresh our knowledge of these.

The derivative of a complex valued function of complex variables

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Let $z = x + iy$, and $f(z) = u(z) + iv(z) = u(x, y) + iv(x, y)$.

x and y real valued. u and v are real valued functions of the real valued variables x and y.

One can show that:

1) If the derivative of $f(z)$ exists at a point z then the partial derivatives of u and v exist at that point and obey the following conditions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{The } \textit{Cauchy-Riemann} \text{ conditions}$$

2) Let u and v be real and single valued functions of x and y which, together with their partial derivatives of the first order, are continuous at a point. If those partial derivatives satisfy the Cauchy-Riemann conditions at that point, then the derivative of f exists at that point.

Analytic Functions

A function $f(z)$ is analytic at a point z_0 if its derivative $f'(z)$ exists not only at z_0 but at every point z in a neighborhood of z_0.

One can show that if $f(z)$ is analytic the partial derivatives of u and v of all orders exist and are continuous functions of x and y.

Thus we have

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y}=\frac{\partial^2 v}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)=-\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=-\frac{\partial^2 u}{\partial y^2}$$

or

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

and in an analogous way we get

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

Thus both $u(x,y)$ and $v(x,y)$ satisfy Laplace's equation.

Harmonic functions

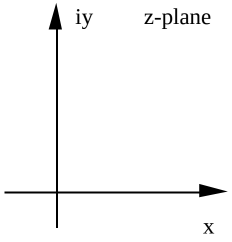
Any function that has continuous partial derivatives of the second order and that satisfies the Laplace's equation is called a Harmonic function.

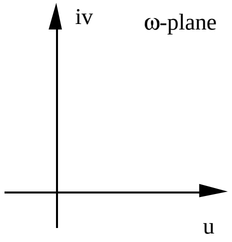
Thus both the real part, u, and imaginary part, v, of f are harmonic functions.

If the function $f = u+\mathrm{i}v$ is analytic, u and v are conjugate harmonic functions. Given one of two harmonic functions, the Cauchy-Riemann equations can be used to find the other.

That u and v are harmonic functions can be used to find the potential. Both u and v are candidates for the potential since they satisfy Laplace's equation. Alternatively the function f can be viewed as a change of variables, a transformation from the complex z-plane to the complex ω-plane.

$$z = x + iy \quad ;$$
$$\omega = u + iv$$





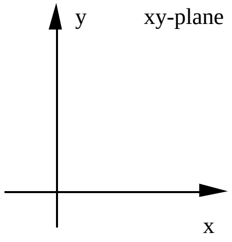
One can show that if the function f is analytic at a point $z=z_0$, where $f'(z_0)\neq 0$, there exists a neighborhood of the point ω_0 in the ω-plane in which the function

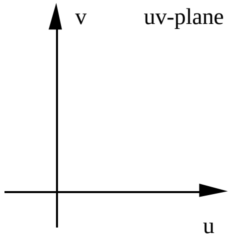
$$\omega = f(z)$$

has a unique inverse

$$z = F(\omega)$$

The functions f and F defines a change of variables from (x,y) to (u,v) and from (u,v) to (x,y), respectively.





When this change of basis is defined through an analytic function all curves in the xy-plane that cross each other at an angle are mapped into curves in the uv-plane that cross each other at exactly the same angle. This is why the mapping is called conformal. The curves can be the boundaries or equi-potential curves, electric field lines or any curves.

In particular since the set of curves that are horizontal in the xy-plane are perpendicular to the set of curves that are vertical the two sets will be mapped into perpendicular sets in the *uv*-plane. Of course the opposite is also true. To be noted is that in points where *f* or its inverse is not analytical the transformation is not conformal.

Transformation of boundary conditions.

Two types of boundary conditions remain unchanged by the transformation and problems with these boundary conditions are best suited for conformal mapping. The types are:

1) The value of a harmonic function is constant on the boundary.
(one type of Dirichlet problem)

2) The normal derivative of a harmonic function is zero on the boundary.
(one type of Neumann problem)

(The conditions are that *f* and its inverse are analytical)

Further properties

Let the real part of the function *f*(*x*,*y*) be the actual potential of our problem. Then the imaginary part is called the *stream function*. Its level curves are called *stream lines*. The name comes from the analogy in fluid flow.

$$f(x,y) = u(x,y) + iv(x,y) = \Phi(x,y) + i\Psi(x,y)$$

In electrostatic problems these stream lines are parallel to the electric field. We furthermore have

$$\mathbf{E}(z) = E_x(x,y) + iE_y(x,y) = -\frac{df^*(z)}{dz}$$

or

$$E_x(x,y) = -\frac{\partial u(x,y)}{\partial x}$$
$$E_y(x,y) = -\frac{\partial u(x,y)}{\partial y}$$

Example 1: The potential from a line charge, of charge density σ_l, along the *z*-axis is $\Phi(r) = -2\sigma_l \ln(r) = -2\sigma_l \,\mathrm{Re}[\ln(z)]$. The electric field calculated in the two different ways is:

$$-\left\{\frac{d}{dz}\left[-2\sigma_l \ln(z)\right]\right\}^* = 2\sigma_l \frac{1}{z^*} = 2\sigma_l \frac{z}{|z|^2} = 2\sigma_l\left[\frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}\right]$$

$$E_x = 2\sigma_l \frac{x}{x^2+y^2} \quad ; \quad E_y = 2\sigma_l \frac{y}{x^2+y^2}$$

and

$$E_x(x,y) = -\frac{\partial u(x,y)}{\partial x} = -\left\{-2\sigma_l \,\partial \ln\left(\sqrt{x^2+y^2}\right) \ \partial x\right\} = 2\sigma_l \frac{x}{x^2+y^2}$$
$$E_y(x,y) = -\frac{\partial u(x,y)}{\partial y} = -\left\{-2\sigma_l \,\partial \ln\left(\sqrt{x^2+y^2}\right) \ \partial y\right\} = 2\sigma_l \frac{y}{x^2+y^2}$$

Example 2: The capacitance between two conductors

If Q is the electrical charge on either conductor and ΔV is the difference in potential between one conductor and the other, then the capacitance C is defined to be

$$C = \frac{|Q|}{|\Delta V|}$$

In two-dimensional problems we calculate the capacitance per unit length, c, whose cross section is typically displayed in the complex plane. Let in this case Q_{L} be the charge per unit length of one of the conductors. Then we have

$$c = \frac{|Q_L|}{|\Delta V|}$$

Theorem: The electrical charge per unit length on a conductor that belongs to a charged two-dimensional configuration of conductors is

$$Q_L = \varepsilon \Delta \Psi(z)$$

where ε is the permittivity, or dielectric constant, of the surrounding material and $\Delta\Psi(z)$ is the decrement (initial value minus final value) of the stream function as we proceed in the positive direction once around the boundary of the cross section of the conductor in the complex plane.

Usually $\Psi(z)$ will be a multivalued function defined by means of a branch cut. Thus $\Psi(z)$ does not return to its original value when we encircle the conductor, and thus $\Delta\Psi \neq 0$. We get:

$$c = \varepsilon \frac{|\Delta\Psi|}{|\Delta V|}$$

Theorem: The capacitance of a two-dimensional system of conductors is unaffected by a conformal transformation of its cross section.

EXAMPLES

We will just give a few examples of conformal mapping.

The Linear Fractional Transformation (Möbius transformation)

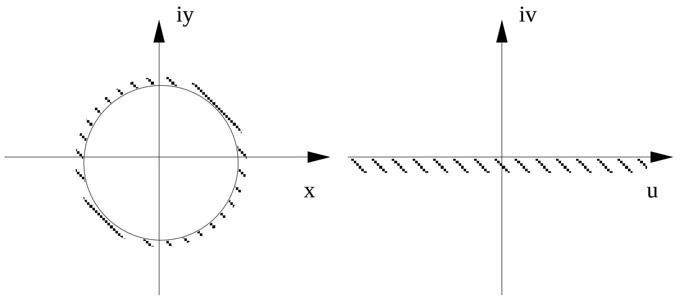
$$\omega = \frac{az+b}{cz+d} \quad , \quad (ad - bc \neq 0) \quad ;$$
$$z = \frac{-d\omega + b}{c\omega - a}$$

where a, b, c and d are complex constants. Each point in the z-plane, except $z = -d/c$, has a unique image point in the ω-plane. And each point in the ω-plane, except $\omega = a/c$, has a unique image point in the z-plane. We may include the points $z = \infty$ and $\omega = \infty$ and then have a one to one correspondence between all points in the z- and ω-planes.

This transformation always transforms circles and lines into circles and lines.

There is just one bilinear transformation that maps three given distinct points z_1, z_2, z_3 into three specified distinct points ω_1, ω_2, ω_3, respectively.

Find one transformation that maps the unit circle and its interior on the real axis and the whole upper half plane.



We let

z = i be mapped into $\omega = \infty$, z = - i be mapped into $\omega = 0$, and
z = 0 be mapped into a point with $v > 0$

The first point guarantees that the circle transforms into a straight line, the second that the line passes through the origin and the third that it maps the interior of the circle to the upper half-plane and not the lower.

The following transformation has the wanted properties:

$$\omega = -i\frac{(z+i)}{(z-i)}$$

The inverse transformation is

$$z = i\frac{\omega - i}{\omega + i}$$

It is useful to introduce the *cross ratio* of four points:

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

Theorem:

The cross ratio is invariant under the möbius transformation :

$$\frac{(\omega_1 - \omega_2)(\omega_3 - \omega_4)}{(\omega_1 - \omega_4)(\omega_3 - \omega_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

This is very useful when we want to find the transformation that maps three specific points, z_1, z_2, z_3 onto three specific images, ω_1, ω_2, ω_3 . Let the forth point be taken as the general point:

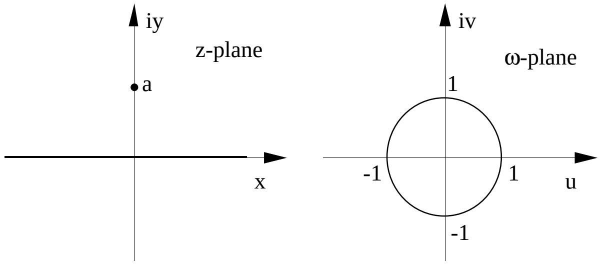
$$\frac{(\omega_1 - \omega_2)(\omega_3 - \omega)}{(\omega_1 - \omega)(\omega_3 - \omega_2)} = \frac{(z_1 - z_2)(z_3 - z)}{(z_1 - z)(z_3 - z_2)}$$

Problem:

Calculate the potential and electric field when a charge distribution

$$\rho(\mathbf{r}) = \sigma_l \delta(x)\delta(y-a)$$

is placed the distance a above a grounded metallic plate in the xz-plane. This is a two-dimensional problem. The charge distribution and boundary condition do not depend on the z-variable.



We will here use the inverse of the transformation, just discussed, and modify it slightly so that the transformation maps the center of the cylinder of radius unity at a distance a above the u-axis (The transformation we had mapped the center a distance unity above the axis). We have

$$\omega = i\frac{z - ia}{z + ia}$$

and

$$z = -ia\frac{(\omega + i)}{(\omega - i)}$$

The problem when the charge distribution is placed along the cylinder axis of a cylinder of radius r_0 is simple to solve. We can use the integral form of Gauss law.

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = 4\pi q_{encl}$$

We have

$$E 2\pi r dz = 4\pi \sigma_l dz$$

and

$$E = \frac{2\sigma_l}{r}$$

The potential is

$$\Phi(r) - \Phi(r_0) = -2\sigma_l \ln\left(\frac{r}{r_0}\right)$$

With our boundary condition and radius we have

$$\Phi(r) = -2\sigma_l \ln(r)$$

or

$$\Phi(u,v) = -\sigma_l \ln\left(u^2 + v^2\right)$$

To find the potential in our given geometry we just use the transformation

$$\Phi(x,y) = -\sigma_l \ln\left[u(x,y)^2 + v(x,y)^2\right]$$

where

$$\omega = i\frac{z - ia}{z + ia} = i\frac{x + i(y-a)}{x + i(y+a)} = i\frac{\left[x + i(y-a)\right]\left[x - i(y+a)\right]}{x^2 + (y+a)^2}$$
$$= \frac{2ax + i\left(x^2 + y^2 - a^2\right)}{x^2 + (y+a)^2}$$

and

$$u(x,y) = \frac{2ax}{x^2 + (y+a)^2} \quad ;$$
$$v(x,y) = \frac{x^2 + y^2 - a^2}{x^2 + (y+a)^2}$$

Our potential is

$$\Phi(x,y) = -\sigma_l \ln\left[\frac{x^4 + y^4 + a^4 + 2x^2y^2 + 2x^2a^2 - 2y^2a^2}{\left[x^2 + (y+a)^2\right]^2}\right]$$
$$= -\sigma_l \ln\left[\frac{\left[x^2 + (y-a)^2\right]}{\left[x^2 + (y+a)^2\right]}\right]$$

The electric field is obtained from this

$$\mathbf{E} = \frac{4\sigma_l a}{\left(x^2 + y^2 + a^2\right)^2 - (2ya)^2}\left(2yx, y^2 - x^2 - a^2\right)$$

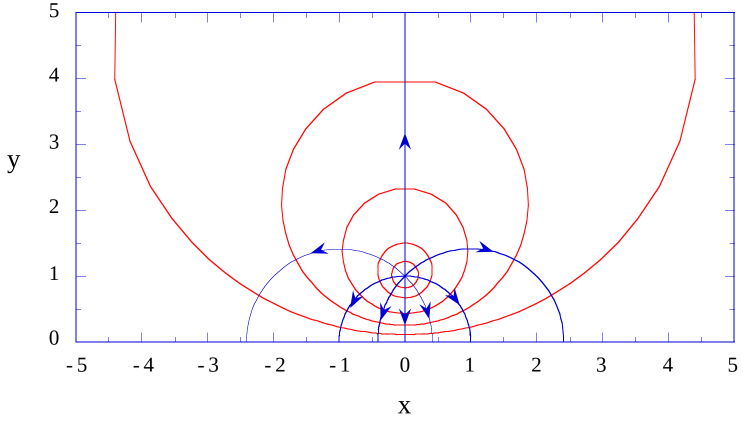
To get level curves for the potential and the electric field lines one could use polar coordinates in the *uv*-plane,

$$u = r\cos(\varphi)\ \ ;$$
$$v = r\sin(\varphi)$$

and let r or φ be constant respectively, and use x and y expressed in u and v.

We have

$$x(u,v) = \frac{2au}{u^2 + (1+v)^2}\ ; \qquad x(r,\varphi) = \frac{2ar\cos(\varphi)}{r^2 + 1 + 2r\sin(\varphi)}\ ;$$
$$\text{or}$$
$$y(u,v) = \frac{a\left(1 - u^2 - v^2\right)}{u^2 + (1+v)^2} \qquad y(r,\varphi) = \frac{a\left(1 - r^2\right)}{r^2 + 1 + 2r\sin(\varphi)}$$



Discussion

We have to be a little careful with the transformations when we have charge distributions. In the problem we just treated we had a grounded plane. This means that the charge of the plane or of the cylinder is equal but with opposite sign to the original charge, the line charge. This means that there is no field outside the mapped regions (outside the cylinder or below the plane) If instead we had given the plane a constant, non-zero potential there had been fields in those regions. In the mapping we bring the point of infinity into a finite position. The electric field has zero divergence in charge free regions and all field lines start or begin at charges. If we have a single charge present the field lines start at the charge and end up at infinity or if the charge is negative the field lines start at infinity and end up at the charge. For a charge distribution with a net charge we may imagine that there is in total the same amount of charge at infinity but with opposite sign. When we now bring this point to a finite position we have to bring this "apparent" charge with it. We should be careful not to include the part containing infinity or the region where infinity has been mapped on a finite point in our region of interest. In the problem we have studied, but with a non zero potential at the boundary, the mapping of the infinity would produce a mirror charge below the *xz*-plane. The fields and potentials above the plane would still be valid, but not the ones below the plane.

Another problem we should address is the potential from a line charge. We have problems finding a suitable constant for the potential so that it is zero at infinity and finite for finite distances. This is because of our mathematical construct of a line charge that extends to infinity at both ends. A real line charge would have finite extension and the potential at large separations from the charge would approach infinity in the same way as that from a point charge. If the net charge is zero the potential approaches infinity as that from a dipole.

Another difficulty is that the geometry of an infinite flat plate is highly unstable. If we bend it a little bit to a cylinder or sphere with almost infinite radius then suddenly one of the sides of the plate becomes the interior of the cylinder or sphere and any excess charge is redistributed to the outer side of the plate and the fields are no longer symmetrical with respect to the plane. The transformation we have used favors this bending.

Another problem is that we have to be careful because the transformation does not conserve area. We have stepped outside the bounds a little bit by using charge densities. Without saying so we scaled the charge distribution in such a way that the total charge was conserved in the transformation.

One may show that the following holds:
If a function *p(x,y)* satisfies *Poisson's equation*

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}=S(x,y)$$

and we use a conformal transformation

$$z=f(\omega)$$

Then the transformed function satisfies the following *Poisson's equation*:

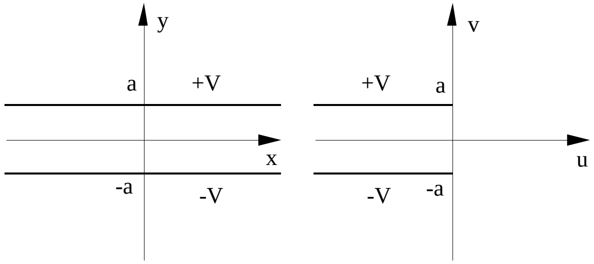
$$\frac{\partial^2 p}{\partial u^2}+\frac{\partial^2 p}{\partial v^2}=S(x(u,v),y(u,v))\left|f'(\omega)\right|^2$$

The electric field cannot be directly transformed either. Instead we have:

$$\left|\text{grad}\,p(x,y)\right|=\left|\text{grad}\,p[x(u,v),y(u,v)]\right|\left|\frac{\partial\omega}{\partial z}\right|$$

We compensate for the change of scale.

Stray fields of a plate capacitor



We want to determine the stray fields and potentials at the edge of a plate capacitor. We start from a geometrical configuration of an ideal plate capacitor where the plates extend to infinity in the two directions. In this case the potential is given by

$$\Phi(y)=\frac{V}{a}y$$

Thus it is apart from a constant factor equal to the variable *y*. We will make a conformal mapping ending up with the configuration given in the *uv*-plane above. The following transformation does the trick:

$$\omega(z)=\tfrac{a}{\pi}\left(1+e^{\frac{\pi}{a}z}\right)+z$$

$$\omega(x\pm ia)=\tfrac{a}{\pi}\left(1-e^{\frac{\pi}{a}x}\right)+x\pm ia$$

$$v(x\pm ia)=\pm a$$
$$u(x\pm ia)=\tfrac{a}{\pi}\left(1-e^{\frac{\pi}{a}x}\right)+x$$
$$u(x\pm ia)_{\max}=u(\pm ia)=0$$

The plates in the *xy*-plane have been folded back on themselves in the *uv*-plane. The line $y=a$ is mapped onto $v=a$, $u\le 0$, $y=-a$ onto $v=-a$, $u\le 0$ and the stripe $-a\le y\le a$ is mapped onto the ω-plane.

We are now ready to find the potential in the new configuration:

$$\Phi(u,v) = \frac{V}{a} y(u,v)$$

It is however difficult to find the inverse to

$$f(z) = \tfrac{a}{\pi}\left(1 + e^{\frac{\pi}{a}z}\right) + z$$

We will be content to find the level curves for the potential, the equi-potential curves, and the electric field lines.

Since

$$z = x + iy = F(\omega)$$

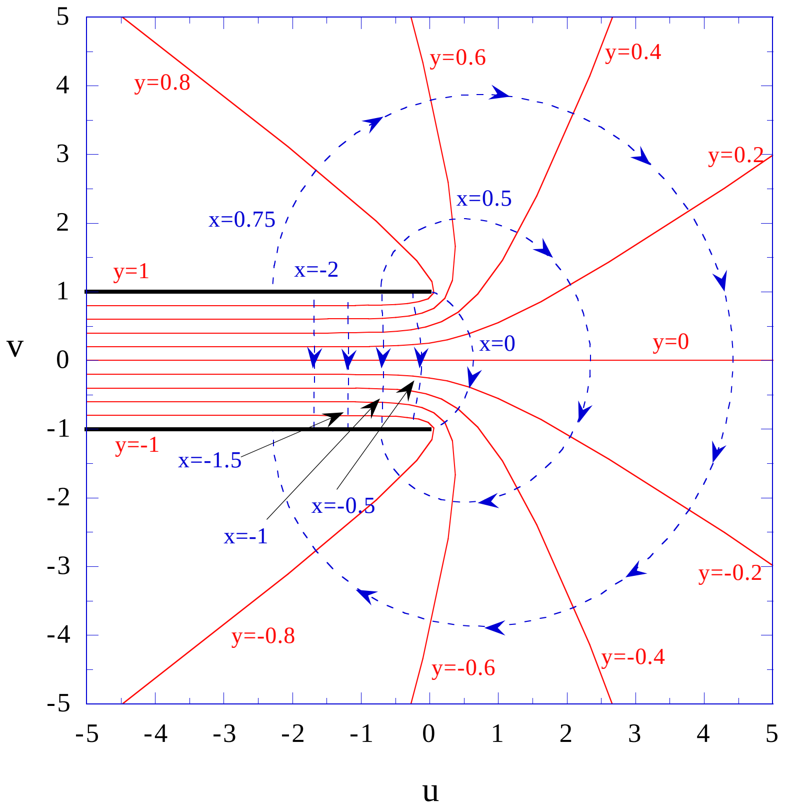
and if the function F is analytic the functions $x(u,v)$ and $y(u,v)$ are conjugate harmonic functions. Their level curves are orthogonal. Since the electric field lines are orthogonal to the equi-potential lines the level curves, $x(u,v) = c$, may represent the field lines.

Now,

$$\begin{aligned} u(x,y) &= \mathrm{Re}\left[\tfrac{a}{\pi}\left(1 + e^{\frac{\pi}{a}(x+iy)}\right) + x + iy\right] \\ &= \tfrac{a}{\pi} + x + \tfrac{a}{\pi} e^{\frac{\pi x}{a}} \cos\left(\tfrac{\pi}{a} y\right) \end{aligned}$$

and

$$\begin{aligned} v(x,y) &= \mathrm{Im}\left[\tfrac{a}{\pi}\left(1 + e^{\frac{\pi}{a}(x+iy)}\right) + x + iy\right] \\ &= y + \tfrac{a}{\pi} e^{\frac{\pi x}{a}} \sin\left(\tfrac{\pi}{a} y\right) \end{aligned}$$

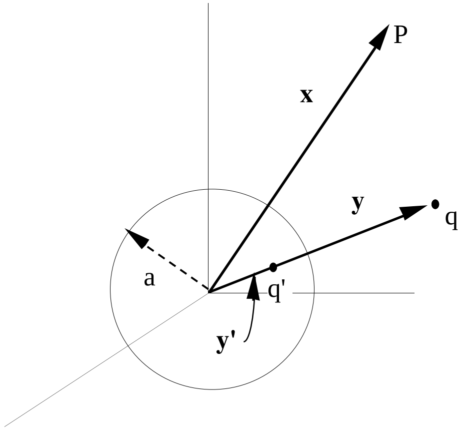


METHOD OF IMAGES

Under favorable conditions it is possible to infer from the geometry of the situation that a small number of suitable placed charges of appropriate magnitudes, external to the region of interest, can simulate the required boundary conditions.

Examples are:
one or more charges on one side of a plane boundary,
one or more charges outside a spherical boundary,
one or more charges inside a spherical boundary,

Point charge in the presence of a grounded conducting sphere.



We have q outside the sphere, a distance y from the origin. We will show that it is enough to place one image charge q' inside the sphere a distance y' from the origin to fulfil the boundary condition $\Phi = 0$ at the spherical surface. From symmetry considerations this image charge must lie on the line connecting the origin and the position of q.

The potential outside the spherical surface from the two charges is:

$$\Phi(\mathbf{x}) = \frac{q}{|\mathbf{x}-\mathbf{y}|} + \frac{q'}{|\mathbf{x}-\mathbf{y}'|}$$
$$= \frac{q}{|x\mathbf{n} - y\mathbf{n}'|} + \frac{q'}{|x\mathbf{n} - y'\,\mathbf{n}'|}$$

On the boundary it is

$$\Phi(a) = \frac{q}{a\left|\mathbf{n} - \frac{y}{a}\mathbf{n}'\right|} + \frac{q'}{y'\left|\mathbf{n}' - \frac{a}{y'}\mathbf{n}\right|}$$

We see that with the choice

$$\frac{q}{a} = -\frac{q'}{y'},\ \frac{y}{a} = \frac{a}{y'}$$

makes the potential vanish on the boundary.

This means

$$\boxed{q' = -\frac{a}{y}q,\ \ y' = \frac{a^2}{y}}$$

Since the solution to Laplace's equation outside the sphere fulfilling the boundary conditions is unique, we have found the potential outside the sphere:

$$\Phi(\mathbf{x}) = \frac{q}{|\mathbf{x}-\mathbf{y}|} - \frac{aq}{y\left|\mathbf{x} - \frac{a^2}{y^2}\mathbf{y}\right|} \quad ; \quad x \geq a$$

We have replaced the conducting sphere with the induced charge density with a single image charge. The two charges q and q' produce the same potential for x ≥ a.

Since we have the correct potential we may calculate the induced charge density on the sphere of the real problem:

$$\sigma(\theta,\varphi) = -\frac{1}{4\pi}\frac{\partial\Phi(x,\theta,\varphi)}{\partial x}\bigg|_{x=a}$$

According to Gauss' law the total induced charge must be equal to the image charge, and it is.

The induced force on the charge q from the sphere can be written down directly using Coulomb's law and the image charge:

$$\begin{aligned}\mathbf{F} &= \frac{qq'}{(y-y')^2}\frac{\mathbf{y}}{y}\\ &= -\left(\frac{a}{y}\right)\frac{q^2}{\left(y-\frac{a^2}{y}\right)^2}\frac{\mathbf{y}}{y}\\ &= -\left(\frac{q}{a}\right)^2\left(\frac{a}{y}\right)^3\left[1-\left(\frac{a}{y}\right)^2\right]^{-2}\frac{\mathbf{y}}{y}\end{aligned}$$

Alternatively the force can be found as minus the force from q acting on the surface of the sphere.

Discussion: Note that the force is attractive at all separations between the charge and the sphere. The induced charge density has opposite sign compared to q all over the sphere.

Point charge in the presence of a charged, insulated, conducting sphere.

$$\Phi(\mathbf{x}) = \frac{q}{|\mathbf{x}-\mathbf{y}|} - \frac{aq}{y\left|\mathbf{x}-\frac{a^2}{y^2}\mathbf{y}\right|} + \frac{Q+\frac{a}{y}q}{|\mathbf{x}|} \quad ; \quad x \ge a$$

The charge Q-q' will distribute itself uniformly over the sphere.

$$\mathbf{F} = \frac{q}{y^2}\left[Q - \frac{qa^3\left(2y^2-a^2\right)}{y\left(y^2-a^2\right)^2}\right]\frac{\mathbf{y}}{y}$$

Discussion: Here the force can be repulsive at large separations if the net charge of the sphere, Q, has the same sign as the charge q. If $Q = 0$ the induced charge density has the same sign as q on the backside of the sphere.

Point charge in the presence of a conducting sphere at fixed potential.

$$\Phi(\mathbf{x}) = \frac{q}{|\mathbf{x}-\mathbf{y}|} - \frac{aq}{y\left|\mathbf{x}-\frac{a^2}{y^2}\mathbf{y}\right|} + \frac{Va}{|\mathbf{x}|} \quad ; \quad x \ge a$$

$$\mathbf{F} = \frac{q}{y^2}\left[Va - \frac{qay^3}{\left(y^2-a^2\right)^2}\right]\frac{\mathbf{y}}{y}$$

Discussion: We have now obtained the induced force between the charge and the conducting sphere for different boundary conditions. If we want to calculate the energy of the system, the work needed to bring the charge from infinity to its position near the sphere, how should we do?

One way is to perform the calculation:

$$W = -\int_{\infty}^{\mathbf{y}} \mathbf{F}(\mathbf{r}) d\mathbf{r} = \int_{y}^{\infty} F(r) dr$$

Can we put $W(y) = q\Phi(y)$?

The answer is no! First we have to neglect the first term in the expressions for the potential in our examples. This term represents the self interaction of the charge itself. The last term in the the last example is O.K. to use as it is. The other terms are induced and should be multiplied with 1/2.
An alternative way to treat the problem is to study the energy density in the field. Using the expression q times the potential corresponds to integrating the energy density over the whole space using the expression valid only outside the sphere. Thus we also include the contribution inside the sphere. This contribution is not real and should be omitted. This part contains half the energy. Just integrating outside the sphere gives the right contribution.

All what we have derived above is also valid for *charge q inside the sphere.* The only difference is that now the expression for the potential is valid only inside the sphere and the total induced surface charge density is $-q$, not q'.
The expression for the surface charge density should be multiplied by -1 since the surface normal now points inwards.

Conducting sphere in a uniform electric field by method of images.

We study a conducting sphere of radius a in a uniform electric field E_0.

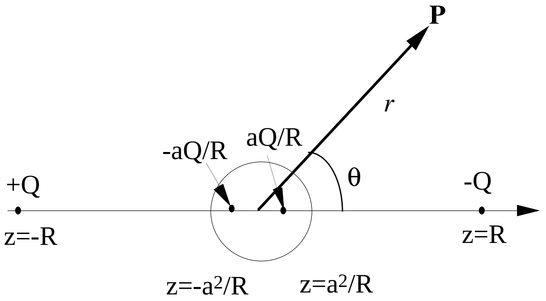
We may view the uniform electric field as originating from two charges of opposite sign and on opposite side of the sphere at infinite separation. The electric field is then

$$E_0 = \frac{2Q}{R^2}$$

or

$$Q = \frac{E_0 R^2}{2}$$

where R will go to infinity in the end



$$\Phi(r,\theta) = -E_0 r\cos\theta + \frac{aQ/R}{\sqrt{(r\sin\theta)^2 + \left(r\cos\theta - a^2/R\right)^2}} + \frac{-aQ/R}{\sqrt{(r\sin\theta)^2 + \left(r\cos\theta + a^2/R\right)^2}}$$

Letting R go to infinity:

$$
\begin{aligned}
\Phi(r,\theta) &\to -E_0 r\cos\theta + \frac{aE_0R}{2}\frac{1}{\sqrt{r^2+a^4/R^2-2a^2r\cos\theta/R}}\\
&\quad -\frac{aE_0R}{2}\frac{1}{\sqrt{r^2+a^4/R^2+2a^2r\cos\theta/R}}\\
&\to -E_0 r\cos\theta + \frac{aE_0R}{2r}\left[\left(1+a^2\cos\theta/rR\right)-\left(1-a^2\cos\theta/rR\right)\right]\\
&\to -E_0 r\cos\theta + \frac{aE_0R}{2r}2a^2\cos\theta/rR\\
&= -E_0 r\cos\theta + \frac{a^3E_0}{r^2}\cos\theta = -E_0\left(r-\frac{a^3}{r^2}\right)\cos\theta
\end{aligned}
$$

Note that the image charges form a dipole of strength

$$D = \frac{aQ}{R}\frac{2a^2}{R} = \frac{2a^3}{R^2}Q = \frac{2a^3}{R^2}\frac{E_0R^2}{2} = a^3E_0$$

The *polarizability of a metallic sphere* is just a^3. We will see this later.

The surface charge density is

$$\sigma = -\frac{1}{4\pi}\frac{\partial\Phi}{\partial r}\bigg|_{r=a} = \frac{3}{4\pi}E_0\cos\theta$$

Method of inversion.

What we have discussed suggests that there is some sort of equivalence of solutions of potential problems under the reciprocal radius transformation,

$$r \to r' = \frac{a^2}{r}$$

This transformation is called *inversion in a sphere*. The radius of the sphere is called the *radius of inversion* and the center of the sphere, *center of inversion*.

The following is true:

Let $\Phi(r,\theta,\phi)$ be the potential due to a set of point charges q_i at the points (r_i,θ_i,ϕ_i). Then the potential

$$\Phi'(r,\theta,\phi) = \frac{a}{r}\Phi\left(\frac{a^2}{r},\theta,\phi\right)$$

is the potential due to charges,

$$q_i' = \frac{a}{r_i}q_i$$

located at the points $\left(a^2/r_i,\theta_i,\phi_i\right)$.

Discussion: If the original problem is one where there are conducting surfaces held at constant potentials, the inverted surfaces will not in general have fixed potentials. The only exception is if the potential is kept zero.

Another thing to be aware of is that there may appear point charges at the center of inversion after the inversion.

Spheres are spheres or planes after the inversion and planes are mapped into spheres that passes through the center of inversion.

Planar interfaces

The results found so far can easily be used to find the image charges for planar interfaces, just by letting the radius of the sphere go to infinity. Let the charge q be placed to the right of the interface at a distance x. The image charge q' will be placed to the left at $-x'$. This means that

$$y = a + x \quad ;$$
$$y' = a - x'$$

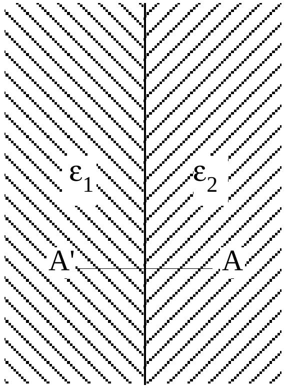
Put this into the relations for q' and y' and let a go to infinity.

$$q' = -\frac{a}{y}q = -\frac{a}{a+x}q \to -q \quad ;$$
$$a - x' = y' = \frac{a^2}{y} = \frac{a^2}{a+x} \to \frac{a^2(1 - x/a)}{a} = a - x$$

Thus the image charge is $-q$ and it is placed at the same distance from the interface as q but on the other side.

METHOD OF IMAGES AT THE BOUNDARY BETWEEN DIELECTRICS

In this case we no longer have the boundary condition that the potential is constant at the interface. We now must use the conditions that the normal components of the **D**-fields and parallel components of the **E**-fields are continuous. We furthermore have fields on both sides of the boundary.



Let us put the charge q in A at the distance d from the interface and place an image charge in A' at the same distance from the interface. The potential from these two charges on the right side is

$$\Phi(\rho, z) = \frac{1}{\varepsilon_2}\left(\frac{q}{\sqrt{\rho^2 + (z-d)^2}} + \frac{q'}{\sqrt{\rho^2 + (z+d)^2}}\right), \quad z > 0$$

To get the potential on the left side we put an effective charge q'' in A to find the potential (replacing q)

$$\Phi(\rho, z) = \frac{1}{\varepsilon_1}\frac{q''}{\sqrt{\rho^2 + (z-d)^2}}, \quad z < 0$$

From this we find

$$\left.\frac{\partial\Phi}{\partial z}\right|_{z=0+}=\frac{1}{\varepsilon_2}\frac{(q-q')d}{\left(\rho^2+d^2\right)^{3/2}}$$

$$\left.\frac{\partial\Phi}{\partial z}\right|_{z=0-}=\frac{1}{\varepsilon_1}\frac{q''d}{\left(\rho^2+d^2\right)^{3/2}}$$

$$\left.\frac{\partial\Phi}{\partial \rho}\right|_{z=0+}=-\frac{1}{\varepsilon_2}\frac{(q+q')\rho}{\left(\rho^2+d^2\right)^{3/2}}$$

$$\left.\frac{\partial\Phi}{\partial \rho}\right|_{z=0-}=-\frac{1}{\varepsilon_1}\frac{q''\rho}{\left(\rho^2+d^2\right)^{3/2}}$$

The boundary conditions give

$$q-q'=q'' \quad ;$$
$$\frac{1}{\varepsilon_2}(q+q')=\frac{1}{\varepsilon_1}q''$$

and

$$q'=-\left(\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right)q \quad ;$$
$$q''=\left(\frac{2\varepsilon_1}{\varepsilon_1+\varepsilon_2}\right)q$$

Discussion: From this we could have obtained the results for a conducting plane if the metal were treated as a dielectric with infinite static dielectric function $\varepsilon_1=\infty$; $q'=-q$ and $q''=2q$. Note that q'' does not contribute to the potential inside the metal since the potential is divided by ε_1 which is infinite. I do not think we can handle the problem with a charge outside a dielectric sphere with a finite number of image charges.