ELECTROMAGNETIC WAVES

Now we will study electromagnetic waves in vacuum or inside a medium, a dielectric. (A metallic system can also be represented as a dielectric but is more complicated due to damping or attenuation for long wave-lengths. We will see that the dielectric function in this case is complex valued).

We start with the more simple case of non conducting media.
With the simplifications that the dielectric function and magnetic permeability are constants with no spatial dependence we get in regions of no free carriers or currents ($\nabla \cdot \mathbf{D} = 0 \rightarrow \nabla \cdot \mathbf{E} = 0$)

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0$$
$$\nabla \times \mathbf{B} - \frac{\varepsilon\mu}{c}\frac{\partial \mathbf{E}}{\partial t} = 0$$

Using the relation

$$\nabla \times (\nabla \times \mathbf{C}) = \nabla(\nabla \cdot \mathbf{C}) - \nabla^2 \mathbf{C}$$

leads to

$$\boxed{\nabla^2 \mathbf{E} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0}$$

$$\boxed{\nabla^2 \mathbf{B} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = 0}$$

The same wave equation for **B** and **E** fields.

The solution to these equations are

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$
$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

where the amplitudes $\mathbf{E}_0$ and $\mathbf{B}_0$ are constant vectors. These represent plane, monochromatic, electromagnetic waves propagating in the direction $\hat{\mathbf{k}}$. The amplitudes depend on one another both as regards the direction and size. (It is assumed that one takes the real value of the fields to represent the real fields.)

To see this we return to what was said in the first lecture about Fourier transforms: Fourier transforming differential equations has the following substitutional effects:

$$\frac{\partial}{\partial t} \rightarrow -i\omega$$
$$\nabla\cdot \;\rightarrow\; i\mathbf{k}\cdot$$
$$\nabla\times \;\rightarrow\; i\mathbf{k}\times$$

Let us now use this in the wave equations:

$$\nabla^2\mathbf{E} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0 \rightarrow -k^2\mathbf{E} + \frac{\varepsilon\mu\omega^2}{c^2}\mathbf{E} = 0$$

or

$$\boxed{k = \sqrt{\varepsilon\mu}\,\frac{\omega}{c} = n\frac{\omega}{c}}$$ *propagation constant* or *wave number*

$$\boxed{n = \sqrt{\varepsilon\mu}}$$ *index of refraction*

The same result is of course obtained if used on the **B** field. Let us apply the procedure on Maxwell's equations

$$
\begin{aligned}
\nabla \cdot \mathbf{E} = 0 &\rightarrow i\mathbf{k} \cdot \mathbf{E} = 0 \\
\nabla \cdot \mathbf{B} = 0 &\rightarrow i\mathbf{k} \cdot \mathbf{B} = 0 \\
\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 &\rightarrow i\mathbf{k} \times \mathbf{E} - i\frac{\omega}{c}\mathbf{B} = 0 \\
\nabla \times \mathbf{B} - \frac{\varepsilon\mu}{c}\frac{\partial \mathbf{E}}{\partial t} = 0 &\rightarrow i\mathbf{k} \times \mathbf{B} + i\frac{\varepsilon\mu\omega}{c}\mathbf{E} = 0
\end{aligned}
$$

The first shows that both **E** and **B** are transverse. The third that also the fields are orthogonal to each other and that **k**, **E**, and **B** form a right handed set of vectors and that

$$
\frac{|\mathbf{B}_0|}{|\mathbf{E}_0|} = n \equiv \sqrt{\varepsilon\mu}
$$

This is also consistent with the last of the equations.

For a travelling wave the electric and magnetic fields oscillate in phase. For a standing wave, which is the superposition of two waves with the same frequency but with **k** vectors in opposite direction, the phase relations are such that the electric and magnetic fields reach their maxima 90 degrees out of phase both in time and in space.

The Poynting vector

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H} = \frac{c}{4\pi\mu}\mathbf{E}\times\mathbf{B}$$

$$= \frac{c}{4\pi\mu}|\mathbf{E}_0||\mathbf{B}_0|\cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t)\hat{\mathbf{k}}$$

$$= \frac{c\sqrt{\varepsilon\mu}}{4\pi\mu}|\mathbf{E}_0|^2\cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t)\hat{\mathbf{k}}$$

$$= \frac{c}{4\pi\eta}|\mathbf{E}_0|^2\cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t)$$

where

$$\eta \equiv \sqrt{\frac{\mu}{\varepsilon}}$$

is the *wave impedance* of the medium.

$$\langle\mathbf{S}\rangle = \frac{c}{8\pi\eta}|\mathbf{E}_0|^2\hat{\mathbf{k}}$$

Convolution integrals

The constitutive relations are for homogeneous and isotropic systems expressed as convolution integrals:

$$\mathbf{D}(\mathbf{r},t) = \int\int d^3r'\, dt' \varepsilon(\mathbf{r}-\mathbf{r}', t-t')\mathbf{E}(\mathbf{r}', t')$$
$$\mathbf{B}(\mathbf{r},t) = \int\int d^3r'\, dt' \mu(\mathbf{r}-\mathbf{r}', t-t')\mathbf{H}(\mathbf{r}', t')$$

Similarly we have:

$$\mathbf{J}^{ind}(\mathbf{r},t) = \int\int d^3r'\, dt' \sigma(\mathbf{r}-\mathbf{r}', t-t')\mathbf{E}(\mathbf{r}', t')$$
$$\mathbf{P}(\mathbf{r},t) = \int\int d^3r'\, dt' \alpha(\mathbf{r}-\mathbf{r}', t-t')\mathbf{E}(\mathbf{r}', t')$$
$$\mathbf{M}(\mathbf{r},t) = \int\int d^3r'\, dt' \chi(\mathbf{r}-\mathbf{r}', t-t')\mathbf{H}(\mathbf{r}', t')$$

where the first equation is Ohm’s law and the other two give the polarization and magnetization of the system in the linear response regime in terms of the dielectric and magnetic polarizabilities, α and χ, respectively. The polarization and magnetization are related to the fields according to:

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$$
$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$$

The convolution integrals have the nice property that their Fourier transforms will be just the product of the two factors in the integrand. Thus, we have

$$\mathbf{D}(\mathbf{q},\omega) = \varepsilon(\mathbf{q},\omega)\mathbf{E}(\mathbf{q},\omega)$$
$$\mathbf{B}(\mathbf{q},\omega) = \mu(\mathbf{q},\omega)\mathbf{H}(\mathbf{q},\omega)$$

and

$$\mathbf{J}^{ind}(\mathbf{q},\omega) = \sigma(\mathbf{q},\omega)\mathbf{E}(\mathbf{q},\omega)$$
$$\mathbf{P}(\mathbf{q},\omega) = \alpha(\mathbf{q},\omega)\mathbf{E}(\mathbf{q},\omega)$$
$$\mathbf{M}(\mathbf{q},\omega) = \chi(\mathbf{q},\omega)\mathbf{H}(\mathbf{q},\omega)$$

These last two equations mean that

$$\varepsilon(\mathbf{q},\omega) = 1 + 4\pi\alpha(\mathbf{q},\omega)$$
$$\mu(\mathbf{q},\omega) = 1 + 4\pi\chi(\mathbf{q},\omega)$$

Another useful fact is that the Fourier transform of the inverse to a correlation function is just unity divided by the Fourier transform of the function itself:

$$\varepsilon^{-1}(\mathbf{q},\omega) = 1/\varepsilon(\mathbf{q},\omega)$$

while

$$\varepsilon^{-1}(\mathbf{r},t) \neq 1/\varepsilon(\mathbf{r},t)$$

COMPLEX REPRESENTATION

We have seen that it is convenient to represent oscillatory functions of time and space by complex exponentials, with the "real" physical function understood to be the real part of the complex function.

It is very useful when we have many functions all with the *same* space-time factor exp$i(\mathbf{k}\cdot\mathbf{r}-\omega t)$ but possibly with different phases. As we have seen the differentiation and integration are reduced to multiplication and division. We put the phase in the complex amplitudes, and the common space-time factor [exp$i(\mathbf{k}\cdot\mathbf{r}-\omega t)$] can usually be cancelled out or ignored. This works since most of the operations are linear, like addition, differentiation, integration and these commutes with taking the real part. So we can perform all operations in the complex plane and take real part at the end.

There is however an important exception: multiplication of two fields like in the calculation of the Poynting vector does not commute with taking the real part. The *product of real parts* of two complex numbers is not in general equal to *the real part of the product*.

In general the result is very involved, but if we are content with the time-average of the product we can by using a trick get away with a simple rule.

Let us have two functions

$$F(t) = F_0 e^{-i\omega t} = |F_0| e^{i\alpha} e^{-i\omega t}$$
$$G(t) = G_0 e^{-i\omega t} = |G_0| e^{i\beta} e^{-i\omega t}$$

Now,

$$\begin{aligned}\left\langle \mathrm{Re}[F(t)]\cdot\mathrm{Re}[G(t)]\right\rangle &= |F_0||G_0|\langle(\cos\omega t\cos\alpha+\sin\omega t\sin\alpha)\\ &\qquad(\cos\omega t\cos\beta+\sin\omega t\sin\beta)\rangle\\ &= |F_0||G_0|\tfrac{1}{2}\cos[\pm(\alpha-\beta)]\\ &= \mathrm{Re}\left[\tfrac{1}{2}F_0\cdot G_0^*\right] = \mathrm{Re}\left[\tfrac{1}{2}F_0^*\cdot G_0\right]\end{aligned}$$

Thus in short hand notation the *time-average product theorem* becomes

$$\boxed{\langle F \cdot G \rangle \Rightarrow \tfrac{1}{2} F_0 \cdot G_0^* = \tfrac{1}{2} F_0^* \cdot G_0}$$

Discussion: Since we have (α-β) in the expression for the time average. the spatial factor exp$i$(**k•r**) in common for F and G is suppressed.

Applying this to Poynting vector gives

$$\langle \mathbf{S} \rangle = \frac{c}{8\pi} \mathbf{E}_0 \times \mathbf{H}_0^* \left[= \frac{c}{8\pi} \mathbf{E}_0^* \times \mathbf{H}_0 \right]$$

where

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \\ \mathbf{H} &= \mathbf{H}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \end{aligned}$$

where $\mathbf{E}_0$ and $\mathbf{H}_0$ are both complex valued vectors.

This can also be applied to the time-averaged energy density for a plane wave in a linear medium:

$$\begin{aligned} \langle \mathscr{E} \rangle &= \frac{1}{16\pi} \left(\mathbf{E}_0 \cdot \mathbf{D}_0^* + \mathbf{H}_0 \cdot \mathbf{B}_0^* \right) \\ &= \frac{1}{16\pi} \left(\varepsilon E_0^2 + \mu H_0^2 \right) = \frac{1}{8\pi} \varepsilon E_0^2 \end{aligned}$$

We see that

$$\langle \mathbf{S} \rangle = \frac{c}{8\pi\eta} E_0^2 \hat{\mathbf{k}} = \frac{c}{\eta\varepsilon} \langle \mathscr{E} \rangle \hat{\mathbf{k}} = \frac{c}{n} \langle \mathscr{E} \rangle \hat{\mathbf{k}} = v \langle \mathscr{E} \rangle \hat{\mathbf{k}}$$