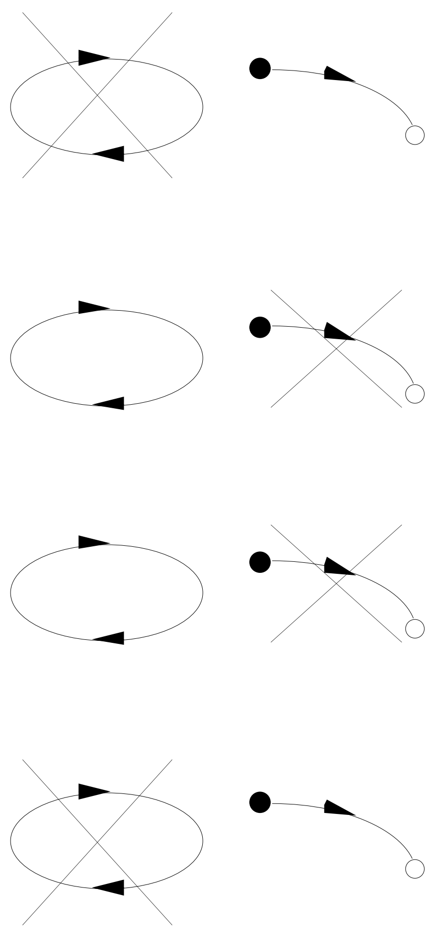
DYNAMIC ELECTROMAGNETISM

Sofar we have only considered static fields produced by static charge and current distributions. When the charge and current distributions are time dependent the fields will be so too and the differential equations describing the fields will now be coupled. We can no longer treat the electric and magnetic fields separately; we must talk about electromagnetic fields.

For static fields in absence of free charge densities and carrier densities we have:

Static fields, no free charges or current



E — **curl E = 0**

D — div **D** = 0

B — div **B** = 0

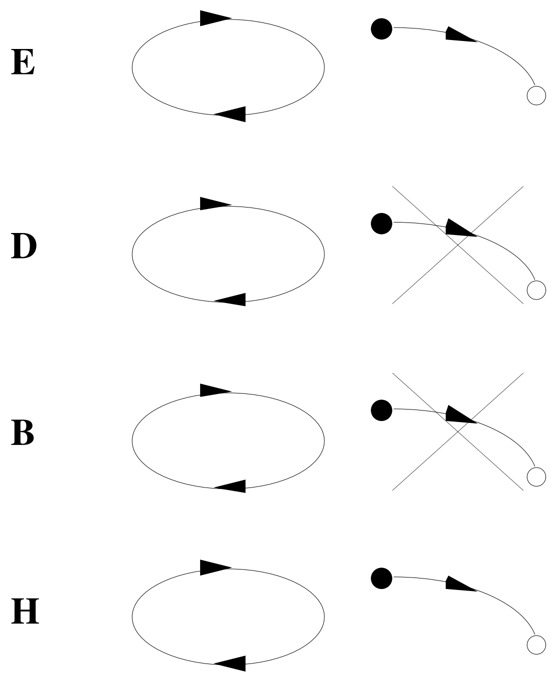
H — **curl H = 0**

(In presence of free charge or current densities the cross for the **D** field or **H** field, respectively should be erased.)

Now, the absence of any free loops for the **E** field means that the field is conservative and one may define a potential. The same is true for the **H** field.

For time dependent fields the situation is different. A time dependent **B** field will produce free **E** field loops and the **E** field is no longer conservative. Similarly a time dependent **D** field will produce free **H** field loops and the **H** field is no longer conservative. This is called *electromagnetic induction*.

Dynamic fields, no free charges or current



Not only the fields are coupled when time dependence is taken into account. Also the charge and current densities are coupled.
This coupling is expressed as the *equation of continuity*. This we treat first.
Then we discuss the coupling between the fields and the new terms in *Maxwell's equations*. Then we introduce the *scalar and vector potentials*. Then the *energy density* in the electromagnetic fields. Finally we touch upon *Maxwell's stress tensor*.

THE EQUATION OF CONTINUITY

A basic assumption in physics is that charge is conserved. It has also been verified experimentally.

Let us study a fixed volume in space. The net outflow of current from this volume has to be equal to minus the rate of change of the charge in the volume:

$$\oint_S \mathbf{J} \cdot \mathbf{n} da = -\frac{dq}{dt} = -\frac{d}{dt}\oint_V \rho dv = -\oint_V \frac{d\rho}{dt} dv$$

The divergence theorem gives

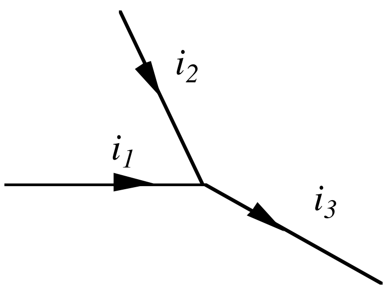
$$\oint_V div\mathbf{J} dv = -\oint_V \frac{d\rho}{dt} dv$$

This holds for an arbitrary volume and hence:

$$\boxed{div\mathbf{J} + \frac{d\rho}{dt} = 0}$$ *Equation of Continuity*

The relation is true for bound quantities, for free quantities and for the total quantities.

This is analogous to Kirchoff's laws in electronic circuit theory.



$$i_3 = i_1 + i_2$$

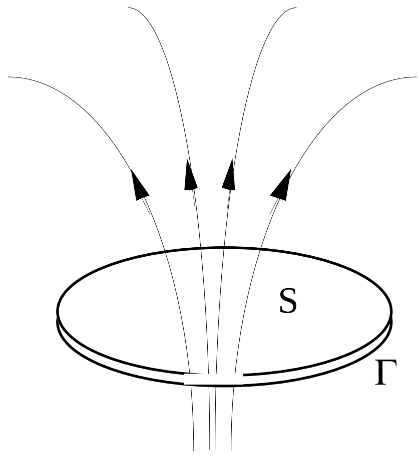
ELECTROMAGNETIC INDUCTION

$$EMF = -\frac{1}{c}\frac{d\Phi_m}{dt}$$ *Faraday's law*

where

$$EMF \equiv \oint_\Gamma \mathbf{E}\cdot d\mathbf{l}$$ *electromotive force*

$$\Phi_m \equiv \int_S \mathbf{B}\cdot \mathbf{n}da$$ *magnetic flux* linking the circuit



Lenz's law:

The current produced by the *EMF* is in a direction that opposes the change of the flux through the circuit.

The rate of change of the flux can be due to the movement of the circuit, the change in the field or due to a combination of both.

If it is due to the change in field we may use Stoke's theorem:

$$\int_S \mathbf{curl}\,\mathbf{E}\cdot \mathbf{n}da = -\frac{1}{c}\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot \mathbf{n}da$$ *S* arbitrary

$$\boxed{\mathbf{curl}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}}$$ *Faraday's law (differential form)*

MAXWELL'S MODIFICATION OF AMPÈRE'S LAW

Ampère's law for steady-state conditions:

$$\mathbf{curl}\,\mathbf{H} = \frac{4\pi}{c}\mathbf{J}_{\text{free}}$$

has to be modified in the case of time dependent fields. This is obvious from taking the divergence of this equation. The left hand side vanishes with the result that the divergence of the current density should vanish in contradiction to the equation of continuity.

Thus we have to add a time derivative of a function on the right hand side:

$$\mathbf{curl}\,\mathbf{H}(\mathbf{r},t) = \frac{4\pi}{c}\mathbf{J}_{\text{free}}(\mathbf{r},t) + \mathbf{F}(\mathbf{r},t)$$

Let us find out what **F**(**r**,t) should be to satisfy the equation of continuity. Take the divergence of the equation,

$$0 = \frac{4\pi}{c}\nabla\cdot\mathbf{J}_{\text{free}}(\mathbf{r},t) + \nabla\cdot\mathbf{F}(\mathbf{r},t)$$

The equation of continuity then gives

$$\nabla\cdot\mathbf{F}(\mathbf{r},t) = \frac{4\pi}{c}\frac{\partial}{\partial t}\rho_{\text{free}}(\mathbf{r},t)$$

and from Gauss' law

$$\nabla\cdot\mathbf{F}(\mathbf{r},t) = \frac{4\pi}{c}\frac{\partial}{\partial t}\frac{1}{4\pi}\nabla\cdot\mathbf{D}(\mathbf{r},t) = \frac{1}{c}\nabla\cdot\left[\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t)\right]$$

The simplest choice is then

$$\mathbf{F}(\mathbf{r},t) = \frac{1}{c}\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t)$$

and

$$\boxed{\mathbf{curl\,H} = \frac{4\pi}{c}\mathbf{J}_{\mathrm{free}} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{D}}$$ *Ampère's law*

$$\mathbf{curl\,H} = \frac{4\pi}{c}\left(\mathbf{J}_{\mathrm{free}} + \mathbf{J}_{\mathrm{d}}\right)$$

where the *displacement current* is

$$\mathbf{J}_{\mathrm{d}} = \frac{1}{4\pi}\frac{\partial}{\partial t}\mathbf{D}$$

$$\boxed{\mathbf{curl\,B} = \frac{4\pi}{c}\left(\mathbf{J}_{\mathrm{free}} + \frac{\partial\mathbf{P}}{\partial t} + c\,\mathbf{curl\,M}\right) + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}}$$ *Maxwell induction*

Thus we had along a loop, real or imagined:

$$\oint_{\Gamma}\mathbf{E}\cdot d\mathbf{l} = -\frac{1}{c}\frac{d}{dt}\int_{S}\mathbf{B}\cdot\mathbf{n}\,da = -\frac{1}{c}\frac{d\Phi_m}{dt}$$

Now, in vacuum we also have

$$\oint_{\Gamma}\mathbf{B}\cdot d\mathbf{l} = \frac{1}{c}\frac{d}{dt}\int_{S}\mathbf{E}\cdot\mathbf{n}\,da = +\frac{1}{c}\frac{d\Phi_e}{dt}$$

MAXWELL'S EQUATIONS

Macroscopic form

$$
\begin{aligned}
\nabla \cdot \mathbf{D} &= 4\pi\rho_f \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0 \\
\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} &= \frac{4\pi}{c}\mathbf{J}_f
\end{aligned}
$$

Microscopic form

$$
\begin{aligned}
\nabla \cdot \mathbf{E} &= 4\pi\rho_t \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0 \\
\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi}{c}\mathbf{J}_t
\end{aligned}
$$

The index f indicates *free* charge or current densities while t denotes *total*. The free densities include external densities and internal unbound densities.
The external charge densities are charges that are not part of the materials but added. It may be impurities in the bulk of a semiconductor or charges at the surface of a charged sample. The external currents can be currents in circuits that we bring into the system in order to study the response to external stimuli.
The internal densities are freely moving charges in metallic systems that can accumulate in regions of the sample and give rise to currents when set in motion.
This division is one of the standard divisions and is used in the text book. I myself prefer to make another division which is useful if one makes a more realistic treatment of metals, taking the finite conductivity and dissipation into account. I will follow the text book in the course in most situations. When I don't I specifically tell so.

If we treat the metals in the same way as we treat dielectrics the macroscopic Maxwell's equations will look the same as before but the densities will now only include external ones. This means that the **D** and **H** fields are now different. The dielectric function and magnetic permeability are different, including the response from the free carriers in the metal. We should remember that these fields are not real, measurable functions. The real functions are the **E** and **B** fields. This becomes obvious here. The **D** and **H** fields depend on our book keeping.

POTENTIALS

Let us now start from the two homogeneous MEs. The first equation,

$$\nabla \cdot \mathbf{B} = \mathbf{0}$$

is fulfilled if we let

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Let us now make use of this relation in the second of the homogeneous MEs

$$\nabla \times \left(\mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

This is fulfilled if what is inside the parentheses is the gradient of a scalar.

Thus for a choice of potentials the fields are obtained from the relations

$$\boxed{\begin{aligned} &\mathbf{E} = -\nabla \Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \\ &\mathbf{B} = \nabla \times \mathbf{A} \end{aligned}}$$

We still have a rather large flexibility to choose the potentials to fit our particular problem, but a change in the scalar potential has to be accompanied by a change in the vector potential and vice versa. Let us assume that we have made a choice of **A** and **Φ**.

Adding a gradient of a scalar function to **A** will not change **B**. We are allowed to change the potentials in the following way:

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\Lambda$$
$$\Phi \to \Phi' = \Phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}$$

These transformations are called *gauge transformations*. The fields, the real quantities, are not changed with such transformations; only the potentials, the auxiliary functions, are.

Let us now turn to the remaining MEs, the inhomogeneous ones. For simplicity we assume vacuum, or use the microscopic version. We have

$$\nabla \cdot \left(\nabla\Phi + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \right) = -4\pi\rho$$
$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c}\frac{\nabla\partial\Phi}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{4\pi}{c}\mathbf{J}$$

which may be rewritten as

$$\nabla^2\Phi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -4\pi\rho$$
$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} + \frac{1}{c}\frac{\nabla\partial\Phi}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{4\pi}{c}\mathbf{J}$$

and after rearrangements in the second one we have

$$\nabla^2\Phi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -4\pi\rho$$
$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} \right) = -\frac{4\pi}{c}\mathbf{J}$$

These are two coupled differential equations; each of them contains both **A** and **Φ**. They can be decoupled by proper gauge transformations. In *Coulomb gauge* or transverse gauge we choose

$$\nabla \cdot \mathbf{A} = 0$$

With this choice the vector potential is purely transverse and the scalar potential satisfies Poisson's equation:

$$\nabla^2 \Phi = -4\pi\rho$$

This means that the scalar potential is *instantaneous*. There are no retardation effects for the scalar potential. So if for example someone on the moon were to play around with some charges, the potential would here on earth change according to this immediately, without any time delay as if information could be transferred faster than with the speed of light.

We will now get rid of the scalar potential from the equation for the vector potential. Let us take the time derivative of Poisson's equation. We have

$$\nabla \cdot \left(\nabla \frac{\partial \Phi}{\partial t} \right) = -4\pi \frac{\partial \rho}{\partial t}$$

The equation of continuity, is valid for the free charge densities and currents, for the induced quantities, and for the total ones. Thus we have

$$\nabla \cdot \left(\nabla \frac{\partial \Phi}{\partial t} \right) = 4\pi \nabla \cdot \mathbf{J}$$

This means that the longitudinal parts of $\nabla \partial\Phi/\partial t$ and $4\pi\mathbf{J}$ are the same. Since the first is purely longitudinal, in the present choice of gauge, we have

$$\nabla \frac{\partial \Phi}{\partial t} = 4\pi \mathbf{J}_L$$

We have divided the external current into transverse and longitudinal parts:

$$\mathbf{J} = \mathbf{J}_\perp + \mathbf{J}_L$$

$$\nabla \cdot \mathbf{J}_\perp = 0$$

$$\nabla \times \mathbf{J}_L = 0$$

Thus the decoupled differential equations for the potentials are

$$\boxed{\begin{aligned}&\nabla^2\Phi = -4\pi\rho\\&\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J}_\perp\end{aligned}}$$

Coulomb gauge

Another common gauge is the *Lorentz gauge*.

In this case we put

$$\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0$$

and get

$$\boxed{\begin{aligned}&\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -4\pi\rho\\&\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J}\end{aligned}}$$

Lorentz gauge

ENERGY IN ELECTROMAGNETIC FIELD

We start from the two curl equations

$$\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$
$$\frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H} - \frac{4\pi}{c}\mathbf{J}_f$$

We multiply the first with **H** and the second by **E** and then add them together:

$$\frac{1}{c}\left(\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} + \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t}\right) = -\frac{4\pi}{c}\mathbf{E}\cdot\mathbf{J}_f - \underbrace{(\mathbf{H}\cdot\nabla\times\mathbf{E} - \mathbf{E}\cdot\nabla\times\mathbf{H})}_{\nabla\cdot(\mathbf{E}\times\mathbf{H})}$$

Assuming linear system so that there are simple constitutive relations between the **H** and **B** fields and between the **D** and **E** fields we have

$$\frac{\partial}{\partial t}\left[\frac{1}{8\pi}(\mathbf{H}\cdot\mathbf{B} + \mathbf{E}\cdot\mathbf{D})\right] = -\mathbf{E}\cdot\mathbf{J}_f - \frac{c}{4\pi}\nabla\cdot(\mathbf{E}\times\mathbf{H})$$
$$= -\mathbf{E}\cdot\mathbf{J}_f - \nabla\cdot\mathbf{S}$$

where we have defined

$$\boxed{\mathbf{S} \equiv \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}} \quad \textit{Poynting vector}$$

Now we integrate the whole expression over a fixed volume and find

$$\int_V \frac{\partial}{\partial t}\left[\frac{1}{8\pi}(\mathbf{H}\cdot\mathbf{B} + \mathbf{E}\cdot\mathbf{D})\right]dv + \int_V \mathbf{E}\cdot\mathbf{J}_f dv + \int_V \nabla\cdot\mathbf{S}dv = 0$$

$$\frac{\partial}{\partial t}\int_V [\mathcal{E}]dv + \int_V \mathbf{E}\cdot\mathbf{J}_f dv + \oint_S \mathbf{S}\cdot\mathbf{n}da = 0$$

The first term is the rate of change of the energy stored in the fields within the volume.
The second terms is the work produced by the fields on the free charges in the

system. The energy transforms into kinetic energy or joule heat.
The last term is the energy leaking out of the system through the surface of the volume per time unit, i.e., the power leaking out of the volume.

Discussion:

What we have said about book keeping has effect here as well. The separation of these terms is not universal.

Had we used the microscopic relations the total current had appeared in the second term, for example. Then the second term had also represented energy stored in the oscillating bound charges (virtual excitations); This energy given away or taken back; It may go in both directions. If the frequency of the field is right real electronic excitations of the atoms may also occur and then there is a real loss of energy in analogy with the Joule heat.

Had we used the other method of book keeping in a metallic system only the external current density had appeared in the second term and the energy lost as Joule heat or for speeding up of the conduction electrons is now stored in the modified energy density of the fields. Also the Poynting vector is modified but the conduction electrons have very small magnetic effects.

THE MAXWELL STRESS TENSOR

Apart from the energy density of the fields there is a momentum density. One can show that:

$$\boxed{\nabla \cdot (-\mathbf{T}) + \frac{\partial \mathbf{g}_{field}}{\partial t} = -\frac{\partial \mathbf{g}_{matter}}{\partial t}}$$

where $\mathbf{g}_{field}$ and $\mathbf{g}_{matter}$ are the respective momentum densities for the fields and matter:

$$\mathbf{g}_{field} \equiv \frac{1}{4\pi c}(\mathbf{E} \times \mathbf{B}) = \frac{1}{c^2}\mathbf{S}$$

and

$$\frac{\partial \mathbf{g}_{matter}}{\partial t} = \rho \mathbf{E} + \frac{1}{c}\mathbf{J} \times \mathbf{B}$$

and $\mathbf{T}$ is the Maxwell stress tensor:

$$T_{ij} = \frac{1}{4\pi}\left[\left(E_i E_j + B_i B_j\right) - \tfrac{1}{2}\left(E^2 + B^2\right)\delta_{ij}\right]$$