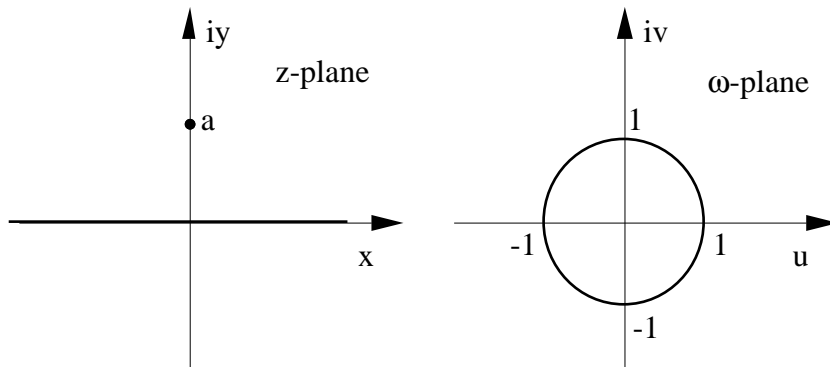


Problem:

Calculate the potential and electric field when a charge distribution

$$\rho(\mathbf{r}) = \sigma_l \delta(x) \delta(y - a)$$

is placed the distance a above a grounded metallic plate in the xz -plane. This is a two-dimensional problem. The charge distribution and boundary condition do not depend on the z -variable.



We will here use the inverse of the transformation, just discussed, and modify it slightly so that the transformation maps the center of the cylinder of radius unity at a distance a above the u -axis (The transformation we had mapped the center a distance unity above the axis). We have

$$\omega = i \frac{z - ia}{z + ia}$$

and

$$z = -ia \frac{(\omega + i)}{(\omega - i)}$$

The problem when the charge distribution is placed along the cylinder axis of a cylinder of radius r_0 is simple to solve. We can use the integral form of Gauss law.

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = 4\pi q_{encl}$$

We have

$$E2\pi r dz = 4\pi\sigma_l dz$$

and

$$E = \frac{2\sigma_l}{r}$$

The potential is

$$\Phi(r) - \Phi(r_0) = -2\sigma_l \ln\left(\frac{r}{r_0}\right)$$

With our boundary condition and radius we have

$$\Phi(r) = -2\sigma_l \ln(r)$$

or

$$\Phi(u, v) = -\sigma_l \ln(u^2 + v^2)$$

To find the potential in our given geometry we just use the transformation

$$\Phi(x, y) = -\sigma_l \ln[u(x, y)^2 + v(x, y)^2]$$

where

$$\begin{aligned}\omega &= i \frac{z - ia}{z + ia} = i \frac{x + i(y - a)}{x + i(y + a)} = i \frac{[x + i(y - a)][x - i(y + a)]}{x^2 + (y + a)^2} \\ &= \frac{2ax + i(x^2 + y^2 - a^2)}{x^2 + (y + a)^2}\end{aligned}$$

and

$$\begin{aligned}u(x, y) &= \frac{2ax}{x^2 + (y + a)^2} ; \\ v(x, y) &= \frac{x^2 + y^2 - a^2}{x^2 + (y + a)^2}\end{aligned}$$

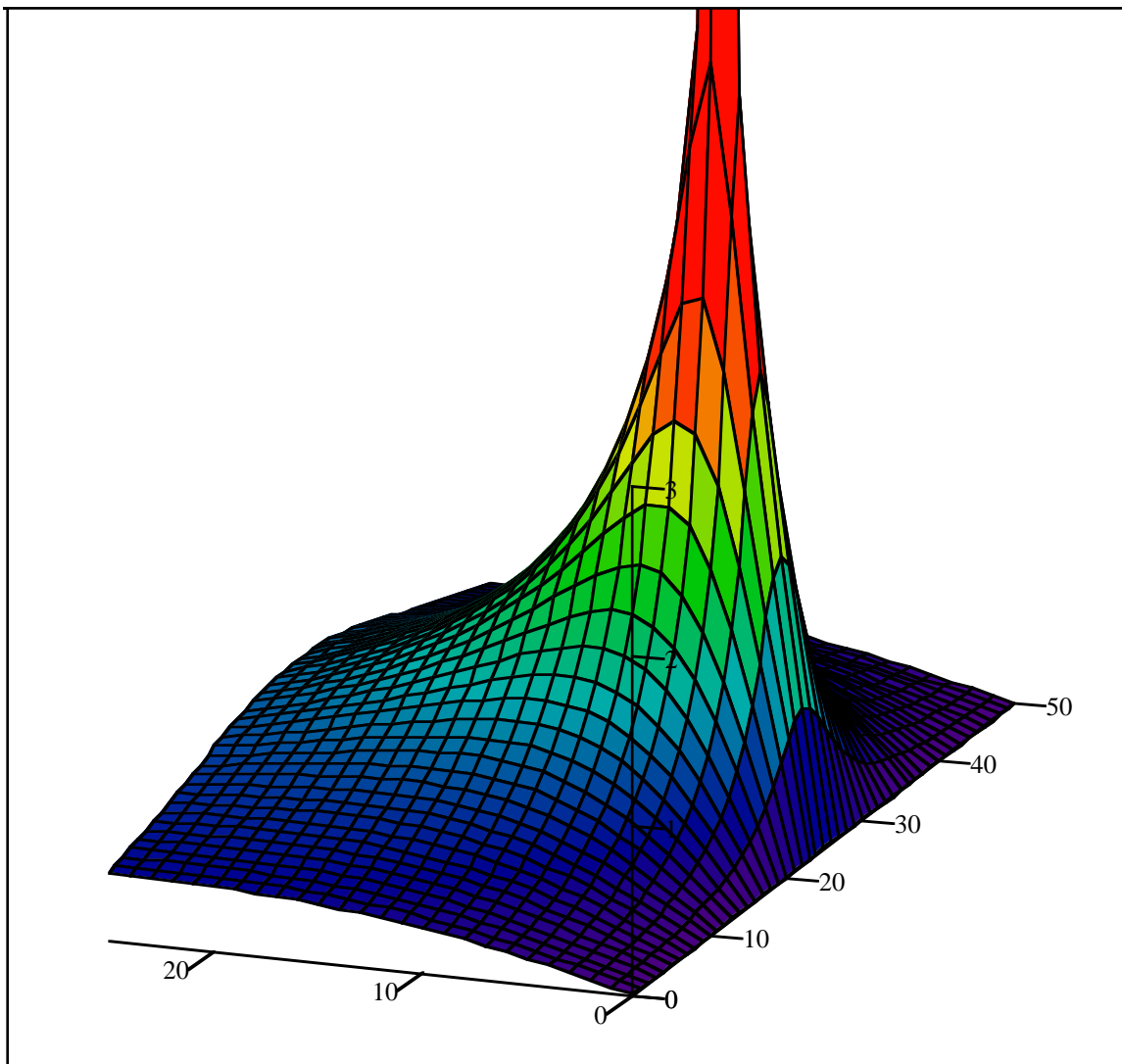
Our potential is

$$\begin{aligned}\Phi(x, y) &= -\sigma_l \ln \left[\frac{x^4 + y^4 + a^4 + 2x^2y^2 + 2x^2a^2 - 2y^2a^2}{[x^2 + (y + a)^2]^2} \right] \\ &= -\sigma_l \ln \left[\frac{[x^2 + (y - a)^2]}{[x^2 + (y + a)^2]} \right]\end{aligned}$$

The electric field is obtained from this

$$\mathbf{E} = \frac{4\sigma_l a}{(x^2 + y^2 + a^2)^2 - (2ya)^2} (2yx, y^2 - x^2 - a^2)$$

The potential:



F

To get level curves for the potential and the electric field lines one could use polar coordinates in the uv -plane,

$$u = r \cos(\varphi) ;$$

$$v = r \sin(\varphi)$$

and let r or φ be constant

respectively, and use x and y expressed in u and v .

We have

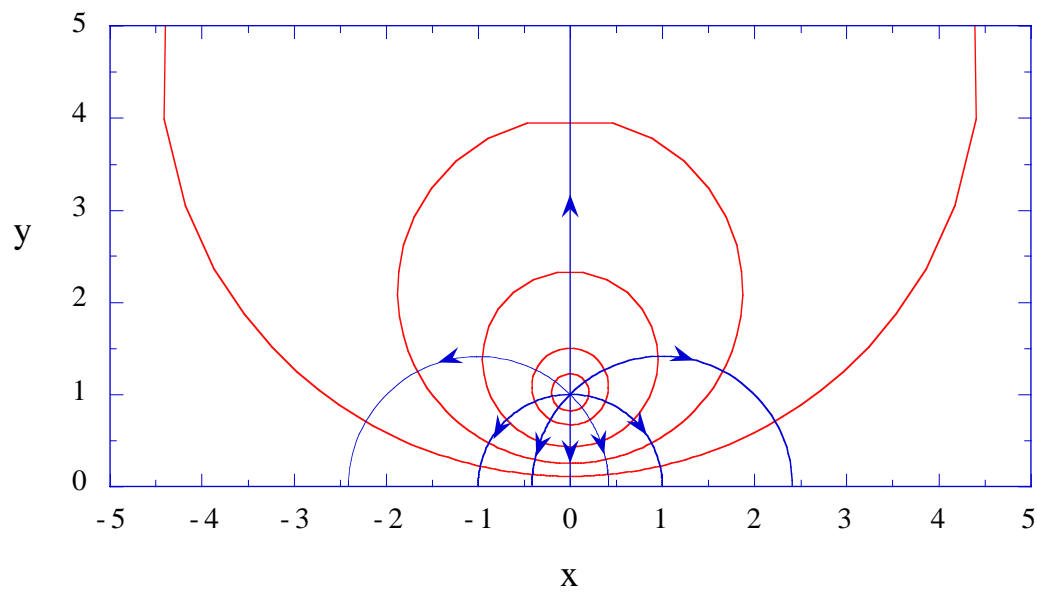
$$x(u, v) = \frac{2au}{u^2 + (1+v)^2} ;$$

$$y(u, v) = \frac{a(1 - u^2 - v^2)}{u^2 + (1+v)^2}$$

or

$$x(r, \varphi) = \frac{2r \cos(\varphi)}{r^2 + 1 + 2r \sin(\varphi)} ;$$

$$y(r, \varphi) = \frac{a(1 - r^2)}{r^2 + 1 + 2r \sin(\varphi)}$$



Discussion

We have to be a little careful with the transformations when we have charge distributions. In the problem we just treated we had a grounded plane. This means that the charge of the plane or of the cylinder is equal but with opposite sign to the original charge, the line charge. This means that there is no field outside the mapped regions (outside the cylinder or below the plane) If instead we had given the plane a constant, non-zero potential there had been fields in those regions. In the mapping we bring the point of infinity into a finite position. The electric field has zero divergence in charge free regions and all field lines start or begin at charges. If we have a single charge present the field lines start at the charge and end up at infinity or if the charge is negative the field lines start at infinity and end up at the charge. For a charge distribution with a net charge we may imagine that there is in total the same amount of charge at infinity but with opposite sign. When we now bring this point to a finite position we have to bring this "apparent" charge with it. We should be careful not to include the part containing infinity or the region where infinity has been mapped on a finite point in our region of interest. In the problem we have studied, but with a non zero potential at the boundary, the mapping of the infinity would produce a mirror charge below the xz -plane. The fields and potentials above the plane would still be valid, but not the ones below the plane.

Another problem we should address is the potential from a line charge. We have problems finding a suitable constant for the potential so that it is zero at infinity and finite for finite distances. This is because of our mathematical construct of a line charge that extends to infinity at both ends. A real line charge would have finite extension and the potential at large separations from the charge would approach infinity in the same way as that from a point charge. If the net charge is zero the potential approaches infinity as that from a dipole.

Another difficulty is that the geometry of an infinite flat plate is highly unstable. If we bend it a little bit to a cylinder or sphere with almost infinite radius then suddenly one of the sides of the plate becomes the interior of the cylinder or sphere and any excess charge is redistributed to the outer side of the plate and the fields are no longer symmetrical with respect to the plane. The transformation we have used favors this bending.

Another problem is that we have to be careful because the transformation does not conserve area. We have stepped outside the bounds a little bit by using charge densities. Without saying so we scaled the charge distribution in such a way that the total charge was conserved in the transformation.

One may show that the following holds:

If a function $p(x,y)$ satisfies *Poisson's equation*

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = S(x,y)$$

and we use a conformal transformation

$$z = f(\omega)$$

Then the transformed function satisfies the following *Poisson's equation*:

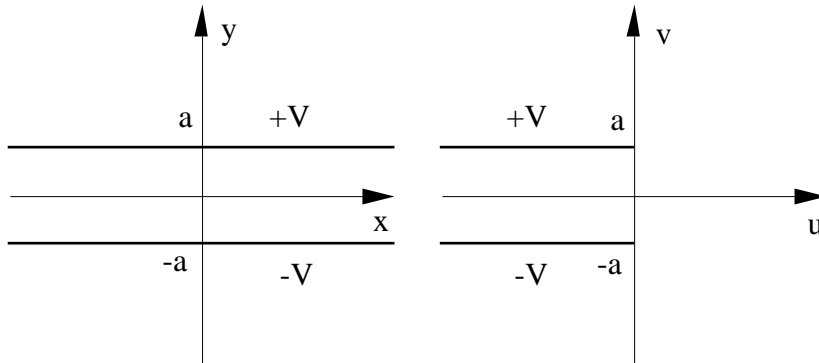
$$\frac{\partial^2 p}{\partial u^2} + \frac{\partial^2 p}{\partial v^2} = S(x(u,v), y(u,v)) |f'(\omega)|^2$$

The electric field cannot be directly transformed either. Instead we have:

$$|\text{grad } p(x,y)| = |\text{grad } p[x(u,v), y(u,v)]| \left| \frac{\partial \omega}{\partial z} \right|$$

We compensate for the change of scale.

Stray fields of a plate capacitor



We want to determine the stray fields and potentials at the edge of a plate capacitor. We start from a geometrical configuration of an ideal plate capacitor where the plates extend to infinity in the two directions. In this case the potential is given by

$$\Phi(y) = \frac{V}{a} y$$

Thus it is apart from a constant factor equal to the variable y . We will make a conformal mapping ending up with the configuration given in the uv -plane above. The following transformation does the trick:

$$\omega(z) = \frac{a}{\pi} \left(1 + e^{\frac{\pi}{a}z} \right) + z$$

$$\omega(x \pm ia) = \frac{a}{\pi} \left(1 - e^{\frac{\pi}{a}x} \right) + x \pm ia$$

$$v(x \pm ia) = \pm a$$

$$u(x \pm ia) = \frac{a}{\pi} \left(1 - e^{\frac{\pi}{a}x} \right) + x$$

$$u(x \pm ia)_{\max} = u(\pm ia) = 0$$

The plates in the xy -plane have been folded back on themselves in the uv -plane. The line $y=a$ is mapped onto $v=a, u \leq 0$, $y=-a$ onto $v=-a, u \leq 0$ and the stripe $-a \leq y \leq a$ is mapped onto the ω -plane.

We are now ready to find the potential in the new configuration:

$$\Phi(u, v) = \frac{V}{a} y(u, v)$$

It is however difficult to find the inverse to

$$f(z) = \frac{a}{\pi} \left(1 + e^{\frac{\pi}{a} z} \right) + z$$

We will be content to find the level curves for the potential, the equi-potential curves, and the electric field lines.

Since

$$z = x + iy = F(\omega)$$

and if the function F is analytic the functions $x(u, v)$ and $y(u, v)$ are conjugate harmonic functions. Their level curves are orthogonal. Since the electric field lines are orthogonal to the equi-potential lines the level curves, $x(u, v) = c$, may represent the field lines.

Now,

$$\begin{aligned} u(x, y) &= \operatorname{Re} \left[\frac{a}{\pi} \left(1 + e^{\frac{\pi}{a}(x+iy)} \right) + x + iy \right] \\ &= \frac{a}{\pi} + x + \frac{a}{\pi} e^{\frac{\pi x}{a}} \cos\left(\frac{\pi}{a} y\right) \end{aligned}$$

and

$$\begin{aligned} v(x, y) &= \operatorname{Im} \left[\frac{a}{\pi} \left(1 + e^{\frac{\pi}{a}(x+iy)} \right) + x + iy \right] \\ &= y + \frac{a}{\pi} e^{\frac{\pi x}{a}} \sin\left(\frac{\pi}{a} y\right) \end{aligned}$$

