LAPLACE'S AND POISSON'S EQUATIONS

$\nabla^2\Phi = -4\pi\rho$ *Poisson's equation*

In regions of no charges the equation turns into:

$\nabla^2\Phi = 0$ *Laplace's equation*

Solutions to Laplace's equation are called *Harmonic Functions*.

Properties of harmonic functions

1) Principle of *superposition* holds

2) A function $\Phi(\mathbf{r})$ that satisfies Laplace's equation in an enclosed volume and satisfies one of the following type of boundary conditions on the enclosing boundary is *unique*.

(a) the value of the function is specified on the whole boundary (Dirichlet condition).

(b) the value of the normal derivative, $\mathbf{n}\cdot\mathbf{grad}\Phi$, is specified on the whole boundary (Neumann condition).

(c) the Φ is specified on part of the boundary and $\mathbf{n}\cdot\mathbf{grad}\Phi$ on the rest.

3) If $\Phi(\mathbf{r})$ satisfies Laplace's equation in a region V, bounded by the surface S, Φ can attain neither a maximum nor a minimum within V.
Extreme values occur only at the surface

Discussion: If the potential is constant on a boundary of a volume not containing any charges the potential has the same constant value within the whole volume.

SOLUTION OF LAPLACE'S EQUATION WITH SEPARATION OF VARIABLES.

There are eleven different coordinate systems in which the Laplace equation is separable. We will here treat the most important ones: the rectangular or cartesian; the spherical; the cylindrical.

The geometry of a typical electrostatic problem is a region free of charges bounded by a surface of a given geometry. It can be of rectangular box type, spherical, cylindrical or of some other type. The standard method then is to choose a coordinate system in which the boundary surface coincides with the surface where one of the coordinates is constant.

In the special case of a 2D configuration, where the bounding surface and the boundary conditions on the surface only depend on two variables, one may use *conformal mapping* to go from one geometrical shape to another.

Other situations are too complicated to solve by these methods. Then one has to rely on purely numerical methods, like solving finite-difference versions of the Laplace's equation, finite element methods (FEM) or some other method.

RECTANGULAR COORDINATES

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

Assume we may write

$$\Phi(x,y,z)=X(x)Y(y)Z(z)$$

$$YZ\frac{d^2X}{dx^2}+XZ\frac{d^2Y}{dy^2}+XY\frac{d^2Z}{dz^2}=0$$

Note that the derivatives are no longer partial.

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0$$

The first term depends on x only, the second on y only and the third on z only. The equation can only be valid if each of the terms is a constant:

$$\frac{1}{X}\frac{d^2X}{dx^2}=\alpha'^2$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2}=\beta'^2$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2}=\gamma'^2$$

$$\alpha'^2+\beta'^2+\gamma'^2=0$$

Since we are considering the electrostatic potential it is real valued. This means that all these squares are real valued, but the last relation shows that the constants themselves cannot all be real valued, neither can they all be imaginary.

We can only have the following cases

a) two real, one imaginary
b) one real, two imaginary
c) one real, one imaginary, one zero
d) three zero

An imaginary separation constant leads to an oscillatory solution while a real valued leads to an exponential.

Let us arbitrarily let α' and β' be imaginary:

$$\alpha'^2 \equiv -\alpha^2$$
$$\beta'^2 \equiv -\beta^2$$
$$\gamma'^2 \equiv \gamma^2$$

α, β and γ are all real valued.

$$\frac{d^2 X}{dx^2} + \alpha^2 X = 0$$
$$\frac{d^2 Y}{dy^2} + \beta^2 Y = 0$$
$$\frac{d^2 Z}{dz^2} - \gamma^2 Z = 0$$
$$\gamma^2 = \alpha^2 + \beta^2 \quad ; \quad \gamma = \sqrt{\alpha^2 + \beta^2}$$

$$X(x) = Ae^{i\alpha x} + Be^{-i\alpha x}$$
$$Y(y) = Ce^{i\beta y} + De^{-i\beta y}$$
$$Z(z) = Ee^{\gamma z} + Fe^{-\gamma z}$$

The complete solution is

$$\Phi(x,y,z) = X(x)Y(y)Z(z)$$

$$\sum_{r,s=1}^{\infty}\left(A_r e^{i\alpha_r x} + B_r e^{-i\alpha_r x}\right)\left(C_s e^{i\beta_s y} + D_s e^{-i\beta_s y}\right)$$

$$\cdot\left(E_{rs} e^{\gamma_{rs} z} + F_{rs} e^{-\gamma_{rs} z}\right)$$

Short hand notation:

$$\boxed{\Phi(x,y,z) \sim e^{\pm i\alpha x} e^{\pm i\beta y} e^{\pm \gamma z}}$$

All the constants will be determined from the boundary conditions of the problem.

SPHERICAL COORDINATES

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

$$\Phi(r,\theta,\varphi) = R(r)P(\theta)Q(\varphi)$$

$$\frac{1}{r^2R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{r^2P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{1}{r^2Q\sin^2\theta}\frac{d^2Q}{d\varphi^2} = 0$$

multiply with $r^2\sin^2\theta$:

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{P}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) = -\frac{1}{Q}\frac{d^2Q}{d\varphi^2}$$

The left-hand side depends only on r and θ, while the right-hand side depends only on ϕ. Thus the two sides must be a constant, m^2.

$$\frac{d^2Q}{d\varphi^2} + m^2Q = 0 \quad ; \quad Q(\varphi) \sim e^{\pm im\varphi} \quad ; \quad m = 0,1,2\ldots$$

Note: If the physical problem limits ϕ to a restricted range m can be a non-integer.

Now we return to the left-hand side and rearrange the terms:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{m^2}{\sin^2\theta}$$

The new left-hand side depends only on r and the right-hand side on only θ. Thus, they must be a constant, $l(l+1)$.

We get

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-l(l+1)R=0$$

and

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]P=0$$

To solve the first, we make the ansatz: $R=Ar^{\alpha}$ and obtain the two solutions r^{l} and $r^{-(l+1)}$. The general solution is then

$$R_l(r)=A_l r^l+B_l\frac{1}{r^{l+1}}$$

For the polar-angle function $P(\theta)$ it is customary to make the substitution

$$\cos\theta\to x\quad;\quad -\frac{1}{\sin\theta}\frac{d}{d\theta}\to\frac{d}{dx}$$

This gives

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP}{dx}\right]+\left[l(l+1)-\frac{m^2}{1-x^2}\right]P=0$$

We will first limit ourselves to axial or azimuthal symmetry.

Axial symmetry

$$\left(1-x^2\right)\frac{d^2P}{dx^2}-2x\frac{dP}{dx}+l(l+1)P=0$$

Legendre's equation

Note that if $x=\pm 1$ are excluded from the problem l may be non-integer.

The solution is the *Legendre polynomial* of order l: $P_l(\cos\theta)$

Thus we have the general solution to Laplace's equation in spherical coordinates for the special case of axial symmetry as:

$$\Phi(r,\theta)=\sum_{l=0}^{\infty}\left[A_l r^l + B_l\frac{1}{r^{l+1}}\right]P_l(\cos\theta)$$

The Legendre polynomials can be obtained from

$$P_l(x)=\frac{1}{2^l l!}\frac{d^l}{dx^l}\left(x^2-1\right)^l$$

Rodrigues' formula

or from the *generating function*

$$F(x,\mu)=\frac{1}{\left(1-2x\mu+\mu^2\right)^{1/2}}=\sum_{l=0}^{\infty}\mu^l P_l(x)$$

or from *recursion relations* such as:

$$(l+1)P_{l+1}(x)=(2l+1)xP_l(x)-lP_{l-1}(x)$$

or

$$\left(1-x^2\right)\frac{dP_l}{dx}=-lxP_l(x)+lP_{l-1}(x)$$

The polynomials form a *complete, orthogonal set* of functions in the domain $-1 \le x \le 1$ $(0 \le \theta \le \pi)$

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x)$$

$$A_l = \frac{2l+1}{2} \int_{-1}^{1} f(x) P_l(x) dx$$

General case, no axial symmetry.

In this case we have in general a non-zero m value and the differential equation for P is more elaborate. The Legendre polynomials are replaced by the *associated Legendre polynomials*, $P_l^m(\cos\theta)$. For a given l-value there are $2l+1$ possible m-values: $m = 0, \pm 1, \pm 2,, \pm 3, ...$

There is a more general *Rodrigues' formula* for these functions:

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}\left(1-x^2\right)^{m/2} \frac{d^{l+m}}{dx^{l+m}}\left(x^2-1\right)^l \quad ; \quad (-l \le m \le +l)$$

For any given m the functions $P_l^m(\cos\theta)$ and $P_{l'}^m(\cos\theta)$ are orthogonal and the associated Legendre polynomials for a fixed m form a complete set of functions in the variable x.

The product of $P_l^m(x)$ and $e^{im\varphi}$ forms a complete set for the expansion of an arbitrary function on the surface of a sphere. These functions are called *spherical harmonics*.

$$\boxed{Y_l^m(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\varphi}}$$

They are orthonormal

$$\begin{aligned}
&\int_{4\pi} Y_l^m(\theta,\varphi)Y_{l'}^{m'}{}^*(\theta,\varphi)d\Omega \\
&\quad = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta Y_l^m(\theta,\varphi)Y_{l'}^{m'}{}^*(\theta,\varphi) = \delta_{ll'}\delta_{mm'}
\end{aligned}$$

$$f(\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_l^m Y_l^m(\theta,\varphi)$$

and

$$C_l^m = \int_{4\pi} f(\theta,\varphi) Y_l^m * (\theta,\varphi) d\Omega$$

The general solution to Laplace's equation in terms of spherical harmonics is

$$\boxed{\Phi(r,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_l^m r^l + B_l^m \frac{1}{r^{l+1}} \right] Y_l^m(\theta,\varphi)}$$

CYLINDRICAL COORDINATES

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

$$\Phi(r,\theta,z) = R(r)Q(\theta)Z(z)$$

$$\frac{1}{rR(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) + \frac{1}{r^2Q(\theta)}\frac{d^2Q(\theta)}{d\theta^2} + \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = 0$$

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) + \frac{r^2}{Z(z)}\frac{d^2Z(z)}{dz^2} = -\frac{1}{Q(\theta)}\frac{d^2Q(\theta)}{d\theta^2} = n^2$$

$$\frac{d^2Q}{d\theta^2} + n^2Q = 0$$

$Q(\theta) \sim e^{\pm in\theta}$; $n = 0,1,2,\ldots$ (n may sometimes be non-integer)

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{n^2}{r^2} = -\frac{1}{Z}\frac{d^2Z}{dz^2} = -k^2$$

$$\frac{d^2Z}{dz^2} - k^2Z = 0$$

$$Z(z) \sim e^{\pm kz}$$

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(k^2r^2 - n^2\right)R = 0$$

Cylindrical symmetry and Cylindrical Harmonics

Then we may let *k* vanish and

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right)-n^2R=0$$

The $n = 0$ term has to be treated separately

$$R_n(r)=\begin{cases}A_0+B_0\ln r, & (n=0)\\ A_nr^n+B_n\dfrac{1}{r^n}, & (n=1,2,3\ldots)\end{cases}$$

$$Q_n(\theta)=\begin{cases}C_0\left[+D_0\theta\right], & (n=0)\\ C_n\cos n\theta+D_n\sin n\theta, & (n=1,2,3\ldots)\end{cases}$$

General solution in cylindrical coordinates with no *z*-dependence.

$$\boxed{\Phi(r,\theta)=A_0+B_0\ln r+\sum_{n=1}^{\infty}\left[A_nr^n+B_n\frac{1}{r^n}\right]\left[C_n\cos n\theta+D_n\sin n\theta\right]}$$

The terms are called *cylindrical harmonics*.

No cylindrical symmetry and Bessel functions.

Now, we have to keep the constant *k* in the differential equation for *R*.

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\left(k^2r^2-n^2\right)R=0$$

To solve this one usually makes the substitution

$$u=kr \quad ; \quad \frac{d}{dr}=k\frac{d}{du}$$

This leads to *Bessel's equation*:

$$\boxed{u^2\frac{d^2R}{du^2}+u\frac{dR}{du}+\left(u^2-n^2\right)R=0}$$

The solution to this equation is the so-called *Bessel function of order n*, $J_n(u)$. $J_{-n}(u)$ is also a solution. These are linearly dependent for integer orders but not for non-integer orders.

One usually introduces another function instead of $J_{-n}(u)$, the so-called *Neumann function* or *Bessel function of the second kind*, $N_n(u)$.

$$N_n(u)=\frac{J_n(u)\cos n\pi-J_{-n}(u)}{\sin n\pi}$$

General solution to Bessel's equation may be written as

$$R_n(kr)=A_nJ_n(kr)+B_nN_n(kr)$$

$J_n(u)$ is regular at origin and at infinity.
$N_n(u)$ is <u>not</u> regular at origin but at infinity.

The general solution to Laplace's equation in cylindrical coordinates can be written as the *Fourier-Bessel expansion*:

$$\Phi(r,\theta,z) \sim \sum_{m,n}\left[A_{mn}J_n\left(k_m r\right)+B_{mn}N_n\left(k_m r\right)\right]e^{\pm in\theta}e^{\pm k_m z}$$

Other useful properties of the Bessel function

Let $k_m\rho$ be the mth root of $J_n(kr)$, i.e., $J_n(k_m\rho) = 0$.

Then $J_n(k_m r)$ form a complete orthogonal set for the expansion of a function of r in the interval $0 \le r \le \rho$.

$$f(r) = \sum_{m=1}^{\infty} D_{mn}J_n\left(k_m r\right) \qquad (\text{for any } n)$$

Fourier-Bessel series

$$D_{mn} = \frac{2}{\rho^2 J_{n+1}^2\left(k_m\rho\right)}\int_0^\infty f(r)J_n\left(k_m r\right)rdr$$

analogous to the Fourier transform.

Discussion: If we had chosen $+k^2$ instead of $-k^2$:

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right)-\frac{n^2}{r^2}=-\frac{1}{Z}\frac{d^2Z}{dz^2}=+k^2$$

The z- dependence had been plane waves instead of exponentials and the r dependence had been found as solutions to the *modified Bessel equation:*

$$\boxed{u^2\frac{d^2R}{du^2}+u\frac{dR}{du}-\left(u^2+n^2\right)R=0}$$

with the modified Bessel functions $I_n(u)$ and $K_n(u)$ as solutions. The first is bounded for small arguments and the second for large.

Thus, an alternative expression for the general solution is

$$\boxed{\Phi(r,\theta,z)\sim\sum_{m.n}\left[A_{mn}I_n\left(k_m r\right)+B_{mn}K_n\left(k_m r\right)\right]e^{\pm in\theta}e^{\pm ik_m z}}$$