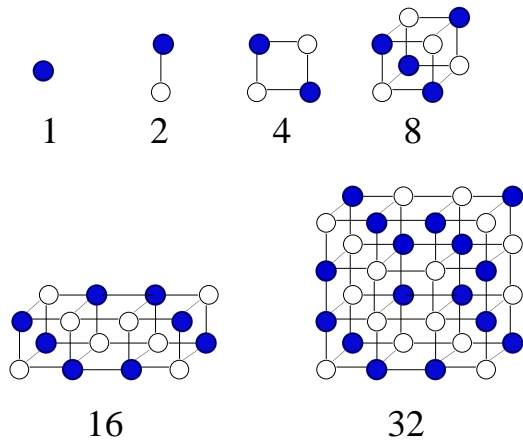
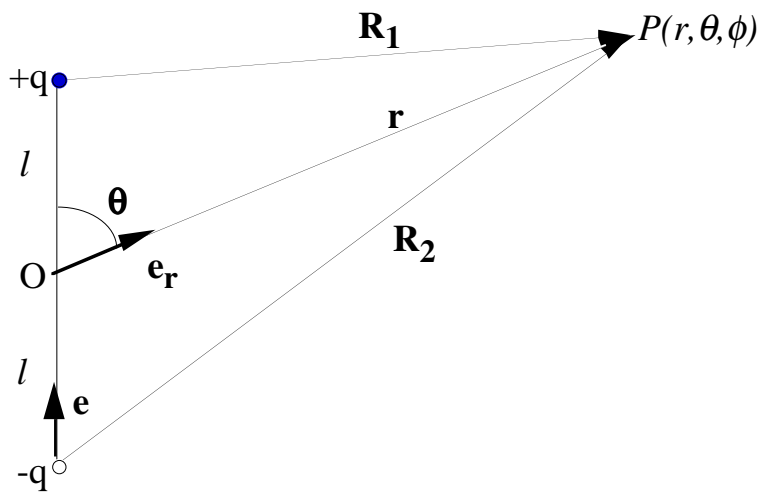


MULTIPOLE FIELDS

Multipoles, 2^l poles. Monopoles, dipoles, quadrupoles, octupoles...



Electric Dipole



The potential at the field point $P(r, \theta, \phi)$ is

$$\Phi(r, \theta, \phi) = q \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

Now

$$R_1^2 = (\mathbf{r} - l\mathbf{e})^2 = r^2 + l^2 - 2rl \cos \theta$$

$$R_2^2 = (\mathbf{r} + l\mathbf{e})^2 = r^2 + l^2 + 2rl \cos \theta$$

and

$$\begin{aligned} \frac{1}{R_1} &= (r^2 + l^2 - 2rl \cos \theta)^{-1/2} = \frac{1}{r} \left(1 + \left(\frac{l}{r}\right)^2 - 2\left(\frac{l}{r}\right) \cos \theta \right)^{-1/2} \\ &= \left| (1+x)^a = 1 + \frac{a}{1}x + \frac{a(a-1)}{1 \cdot 2}x^2 + \frac{a(a-1)(a-2)}{1 \cdot 2 \cdot 3}x^3 + \dots \right| \\ &= \frac{1}{r} \left[1 + \left(\frac{l}{r}\right) \cos \theta + \frac{1}{2} \left(\frac{l}{r}\right)^2 (3 \cos^2 \theta - 1) \right. \\ &\quad \left. + \frac{1}{2} \left(\frac{l}{r}\right)^3 (5 \cos^3 \theta - 3 \cos \theta) + \dots \right] \end{aligned}$$

$$\begin{aligned} \frac{1}{R_2} &= \frac{1}{r} \left[1 - \left(\frac{l}{r}\right) \cos \theta + \frac{1}{2} \left(\frac{l}{r}\right)^2 (3 \cos^2 \theta - 1) \right. \\ &\quad \left. - \frac{1}{2} \left(\frac{l}{r}\right)^3 (5 \cos^3 \theta - 3 \cos \theta) + \dots \right] \end{aligned}$$

The potential becomes *approximately*

$$\Phi(r, \theta, \varphi) \approx 2ql \frac{\cos \theta}{r^2}$$

We have assumed that $l \ll r$.

The electric dipole moment is defined as

$$\mathbf{p} = 2ql\mathbf{e}$$

The lowest order (in l/r) contribution to the potential from the pair of charges is the dipole contribution. An ideal dipole, a *point dipole*, is obtained if we let the separation of the charges go to zero and at the same time let the charges go

to infinity so that ql is kept constant ($2ql=p$). If we do that only the lowest order contribution to the potential survives. *To be noted is that if we keep the separation of the charges finite even if the charges are point charges there are higher order multipole contributions.*

The dipole potential can be written as

$$\Phi = \frac{\mathbf{p} \cdot \mathbf{e}_r}{r^2} = \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}$$

The electric field is

$$\mathbf{E} = -\nabla\Phi$$

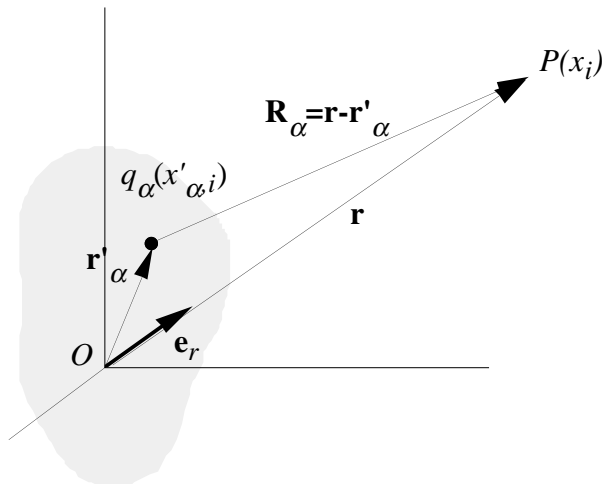
and in spherical coordinates we have

$$E_r = -\frac{\partial\Phi}{\partial r} = 2p \frac{\cos\theta}{r^3}$$

$$E_\theta = -\frac{1}{r} \frac{\partial\Phi}{\partial\theta} = p \frac{\sin\theta}{r^3}$$

$$E_\varphi = -\frac{1}{r\sin\theta} \frac{\partial\Phi}{\partial\varphi} = 0$$

MULTIPOLE EXPANSION



Unprimed \mathbf{r} -vectors represent field points.

Primed represent source points.

Upper case, \mathbf{R} vectors, represent vectors from source points to field points.

Greek subscripts refer to individual charges or particles.

Roman subscripts refer to coordinate axes.

The potential from charge q_α is

$$\Phi_\alpha = \frac{q_\alpha}{R_\alpha}$$

We want to expand $1/R_\alpha$ around the source origin keeping the field point fixed.

The general Taylor expansion is

$$f(\mathbf{r}'_\alpha) = f(0) + \frac{1}{1!} \sum_i x'_{\alpha,i} \left[\frac{\partial f(\mathbf{r}'_\alpha)}{\partial x'_{\alpha,i}} \right]_{\mathbf{r}'_\alpha=0} \\ + \frac{1}{2!} \sum_{i,j} x'_{\alpha,i} x'_{\alpha,j} \left[\frac{\partial^2 f(\mathbf{r}'_\alpha)}{\partial x'_{\alpha,i} \partial x'_{\alpha,j}} \right]_{\mathbf{r}'_\alpha=0} + \dots$$

This gives for the potential from one of the point charges

$$\Phi_\alpha(\mathbf{r}) = \frac{q_\alpha}{r} + q_\alpha \sum_i x'_{\alpha,i} \left[\frac{\partial}{\partial x'_{\alpha,i}} \left(\frac{1}{R_\alpha} \right) \right]_{R_\alpha=r} \\ + \frac{1}{2} q_\alpha \sum_{i,j} x'_{\alpha,i} x'_{\alpha,j} \left[\frac{\partial^2}{\partial x'_{\alpha,i} \partial x'_{\alpha,j}} \left(\frac{1}{R_\alpha} \right) \right]_{R_\alpha=r} + \dots$$

Now since

$$R_\alpha = \sqrt{\sum_i (x_i - x'_{\alpha,i})^2}$$

we have

$$\left[\frac{\partial}{\partial x'_{\alpha,i}} \left(\frac{1}{R_\alpha} \right) \right]_{R_\alpha=r} = - \left[\frac{\partial}{\partial x_i} \left(\frac{1}{R_\alpha} \right) \right]_{R_\alpha=r} = - \frac{\partial}{\partial x_i} \left(\frac{1}{r} \right)$$

This means that

$$\begin{aligned}\Phi_{\alpha}(\mathbf{r}) &= \frac{q_{\alpha}}{r} - q_{\alpha} \sum_i x'_{\alpha,i} \frac{\partial}{\partial x_i} \left(\frac{1}{r} \right) \\ &\quad + \frac{1}{2} q_{\alpha} \sum_{i,j} x'_{\alpha,i} x'_{\alpha,j} \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{1}{r} \right) + \dots\end{aligned}$$

The total potential from all charges may then be written as

$$\Phi(\mathbf{r}) = \sum_{\alpha} \Phi_{\alpha}(\mathbf{r}) = \Phi^{(1)}(\mathbf{r}) + \Phi^{(2)}(\mathbf{r}) + \Phi^{(4)}(\mathbf{r}) + \dots \Phi^{(2^l)}(\mathbf{r}) + \dots$$

where

$$\begin{aligned}\Phi^{(1)}(\mathbf{r}) &= \sum_{\alpha} \frac{q_{\alpha}}{r} = \frac{q}{r} \\ \Phi^{(2)}(\mathbf{r}) &= - \sum_{\alpha} q_{\alpha} \sum_i x'_{\alpha,i} \frac{\partial}{\partial x_i} \left(\frac{1}{r} \right) \\ \Phi^{(4)}(\mathbf{r}) &= \frac{1}{2} \sum_{\alpha} q_{\alpha} \sum_{i,j} x'_{\alpha,i} x'_{\alpha,j} \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{1}{r} \right) \\ \Phi^{(2^l)}(\mathbf{r}) &= \frac{(-1)^l}{l!} \sum_{\alpha} q_{\alpha} \sum_{i,j,\dots,l} x'_{\alpha,i} x'_{\alpha,j} \dots x'_{\alpha,l} \frac{\partial^l}{\partial x_i \partial x_j \dots \partial x_l} \left(\frac{1}{r} \right)\end{aligned}$$

where the first term is the *monopole potential* the second the *dipole potential*, the third the *quadrupole potential*, and so on. The last, general terms, is called the 2^l th multipole potential. To be noted is that each term drops off faster with distance with one power in r than the previous term.

$$\Phi^{(2^l)}(\mathbf{r}) \propto \frac{1}{r^{l+1}}$$

This means that for large distances the lowest order non-vanishing multipole potential dominates.

THE DIPOLE POTENTIAL

$$\begin{aligned}
 \Phi^{(2)}(\mathbf{r}) &= -\sum_{\alpha} q_{\alpha} \sum_i x'_{\alpha,i} \frac{\partial}{\partial x_i} \left(\frac{1}{r} \right) \\
 &= -\sum_{\alpha} q_{\alpha} \mathbf{r}'_{\alpha} \cdot \mathbf{grad} \left(\frac{1}{r} \right) \\
 &= -\mathbf{p} \cdot \mathbf{grad} \left(\frac{1}{r} \right) \\
 &= -\mathbf{p} \cdot \left(-\frac{\mathbf{r}}{r^3} \right)
 \end{aligned}$$

Thus

$$\boxed{\Phi^{(2)}(\mathbf{r}) = \frac{\mathbf{p} \cdot \mathbf{e}_r}{r^2}}$$

where the *dipole moment* of the system is defined as:

$$\boxed{\mathbf{p} = \sum_{\alpha} q_{\alpha} \mathbf{r}'_{\alpha}}$$

In continuous case:

$$\mathbf{p} = \int d^3 r' \rho(\mathbf{r}') \mathbf{r}'$$

The electric dipole field vector is

$$\begin{aligned}
 \mathbf{E}^{(2)}(\mathbf{r}) &= -\mathbf{grad}\Phi^{(2)}(\mathbf{r}) \\
 &= -\mathbf{grad}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} \\
 &= -\frac{1}{r^3}\mathbf{grad}(\mathbf{p}\cdot\mathbf{r}) - (\mathbf{p}\cdot\mathbf{r})\mathbf{grad}\frac{1}{r^3} \\
 &= -\frac{1}{r^3}\mathbf{p} + (\mathbf{p}\cdot\mathbf{r})\frac{3\mathbf{r}}{r^5} \\
 &= \frac{1}{r^5}\left[3(\mathbf{p}\cdot\mathbf{r})\mathbf{r} - \mathbf{p}r^2\right]
 \end{aligned}$$

This may alternatively be written on the form

$$\mathbf{E} = -\tilde{\phi}\mathbf{p} ; E_i = -\sum_j \phi_{ij} p_j$$

where ϕ is the so-called dipole-dipole tensor with elements:

$$\phi_{ij} = \frac{\delta_{ij}}{r^3} - \frac{3r_i r_j}{r^5}$$

THE QUADRUPOLE POTENTIAL

$$\Phi^{(4)}(\mathbf{r}) = \frac{1}{2} \sum_{\alpha} q_{\alpha} \sum_{i,j} x'_{\alpha,i} x'_{\alpha,j} \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{1}{r} \right)$$

This expression may be used as it stands but we will introduce the quadrupole tensor.

First we note that since $1/r$ is a solution to Laplace's equation we have

$$\sum_i \frac{\partial^2}{\partial x_i^2} \left(\frac{1}{r} \right) = 0, \quad r > 0$$

This may be written as

$$\sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{1}{r} \right) \delta_{ij} = 0, \quad r > 0$$

Since this is a null quantity any constant times this may be added to $\Phi^{(4)}$ without altering its value. We choose the constant to be $-\frac{1}{6} \sum_{\alpha} q_{\alpha} r_{\alpha}^2$ and obtain

$$\Phi^{(4)}(\mathbf{r}) = \frac{1}{6} \sum_{\alpha} q_{\alpha} \sum_{i,j} \left(3x'_{\alpha,i} x'_{\alpha,j} - r_{\alpha}^2 \delta_{ij} \right) \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{1}{r} \right)$$

This may be written as

$$\Phi^{(4)}(\mathbf{r}) = \frac{1}{6} \sum_{i,j} Q_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{1}{r} \right) = \frac{1}{6} \sum_{i,j} Q_{ij} \left(\frac{3x_i x_j - r^2 \delta_{ij}}{r^5} \right) = -\frac{1}{6} \sum_{i,j} Q_{ij} \phi_{ij}$$

where the *quadrupole tensor* $\tilde{\mathbf{Q}}$ has the elements.

$$Q_{ij} = \sum_{\alpha} q_{\alpha} (3x'_{\alpha,i}x'_{\alpha,j} - r'_{\alpha}{}^2 \delta_{ij})$$

This tensor is symmetric which means at most 6 independent elements. Actually there are at most 5 independent elements since one can show that the *trace*, sum of the diagonal elements, is zero.

Also the dipole-dipole tensor is symmetric which means that

$$\begin{aligned} \Phi^{(4)}(\mathbf{r}) &= -\frac{1}{6} \sum_{i,j} Q_{ij} \phi_{ij}(\mathbf{r}) = -\frac{1}{6} \sum_{i,j} Q_{ij} \phi_{ji}(\mathbf{r}) \\ &= -\frac{1}{6} \sum_i [\tilde{\mathbf{Q}} \cdot \tilde{\phi}(\mathbf{r})]_{ii} = -\frac{1}{6} \text{Tr}[\tilde{\mathbf{Q}} \cdot \tilde{\phi}(\mathbf{r})] \end{aligned}$$

If the quadrupole tensor is referred to principal axes it becomes diagonal. This together with the vanishing trace means that there are only 2 independent elements.

In many applications the charge distribution has an axis of symmetry. If we choose this to be x'_3 axis, then $Q_{11} = Q_{22}$. Then there is only one independent element.

$$\tilde{\mathbf{Q}} = \begin{pmatrix} -Q/2 & 0 & 0 \\ 0 & -Q/2 & 0 \\ 0 & 0 & Q \end{pmatrix}$$

For continuous charge distribution:

$$Q = \int_V \rho(\mathbf{r}') (3x_3'^2 - r'^2) dx_1' dx_2' dx_3'$$

With this simple special case the quadrupole potential becomes

$$\Phi^{(4)}(r, \theta) = \frac{1}{4} Q \frac{(3 \cos^2 \theta - 1)}{r^3}$$

Discussion:

The multipole moments in general depend on the position of the origin. However the lowest non-vanishing moment is independent of the position of the origin.