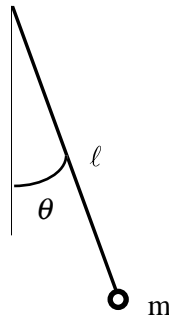


ELECTROMAGNETIC NORMAL MODES AND DISPERSION FORCES.

All systems with interaction of some type have normal modes. One may describe them as solutions in absence of sources; they are excitations of the system that can appear without any external stimuli; they are solutions to the homogeneous versions of the differential equations that describe the system. Before we study the electromagnetic normal modes let us study some mechanical systems. We start with the single pendulum where a mass, m , is attached to a string of length l . This system has one normal mode.



The system is governed by Newton's 2nd law. For small angles this law is reduced to

$$l\ddot{\theta} + g\theta = 0.$$

The Fourier transformed version is

$$\left[l(-i\omega)^2 + g \right] \theta = 0,$$

which has non-trivial solutions if

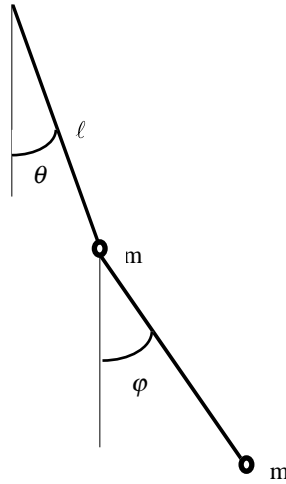
$$l(-i\omega)^2 + g = 0,$$

i.e.,

$$\omega = \sqrt{g/l}.$$

This is the frequency of the normal mode.

Let us now extend this problem to the double pendulum.



For small amplitudes the equations are reduced to

$$2l\ddot{\theta} + 2g\theta + l\ddot{\varphi} = 0$$

$$l\ddot{\theta} + l\ddot{\varphi} + g\varphi = 0.$$

Fourier transformation gives

$$\left(-2l\omega^2 + 2g\right)\theta - l\omega^2\varphi = 0$$

$$-l\omega^2\theta + \left(-l\omega^2 + g\right)\varphi = 0.$$

This system of equations may be written on matrix form,

$$A \begin{pmatrix} \theta \\ \varphi \end{pmatrix} = 0; \quad A = \begin{pmatrix} -2l\omega^2 + 2g & -l\omega^2 \\ -l\omega^2 & -l\omega^2 + g \end{pmatrix}.$$

There are non-trivial solutions if the determinant of the matrix is zero,

$$2\left(-l\omega^2 + g\right)^2 - \left(-l\omega^2\right)^2 = 0,$$

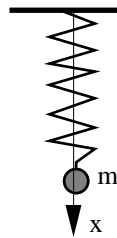
or

$$\sqrt{2}(-l\omega^2 + g) \pm (l\omega^2) = 0,$$

or

$$\omega_1 = \sqrt{(2 + \sqrt{2})g/l}; \quad \omega_2 = \sqrt{(2 - \sqrt{2})g/l}.$$

Thus, there are two normal modes in this system. In the one with larger frequency the two masses move out of phase of each other and in the other they are in phase. Next we study a mass in a spring



If we express x as the displacement from the equilibrium point we have the equation of motion as

$$\ddot{x} + \frac{k}{m}x = 0.$$

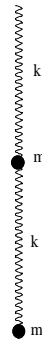
On Fourier transformed form it is

$$(-\omega^2 + k/m)x = 0,$$

with non-trivial solutions for

$$\omega = \sqrt{k/m}.$$

For two masses and two springs we have



$$\omega_{1,0} = \sqrt{k/m} = \omega_0 ;$$

$$\omega_{2,0} = \sqrt{2k/m} = \sqrt{2}\omega_0$$

where the first (second) frequency is the frequency with which the lowest (upper) mass is oscillating if we keep the upper (lower) in fixed position. When we let them both loose we have the two normal modes from the equations

$$m\ddot{z}_1 + 2kz_1 - kz_2 = 0 ;$$

$$m\ddot{z}_2 + kz_2 - kz_1 = 0$$

Fourier transforming gives

$$\begin{pmatrix} -m\omega^2 + 2k & -k \\ -k & -m\omega^2 + k \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$$

and the condition for modes,

$$\begin{vmatrix} -m\omega^2 + 2k & -k \\ -k & -m\omega^2 + k \end{vmatrix} = 0 ,$$

which gives

$$(-m\omega^2 + 2k)(-m\omega^2 + k) - (-k)^2 = 0$$

$$\omega^4 - \omega^2 \left(\frac{3k}{m} \right) + \left(\frac{k}{m} \right)^2 = 0$$

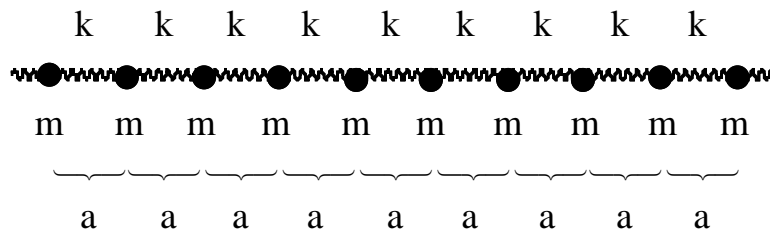
$$\omega^4 - 3\omega^2\omega_0^2 + \omega_0^4 = 0$$

and finally

$$\omega_1 = \omega_0 \sqrt{\frac{3 - \sqrt{5}}{2}} ;$$

$$\omega_2 = \omega_0 \sqrt{\frac{3 + \sqrt{5}}{2}}$$

Now we study many masses attached to strings as in the figure



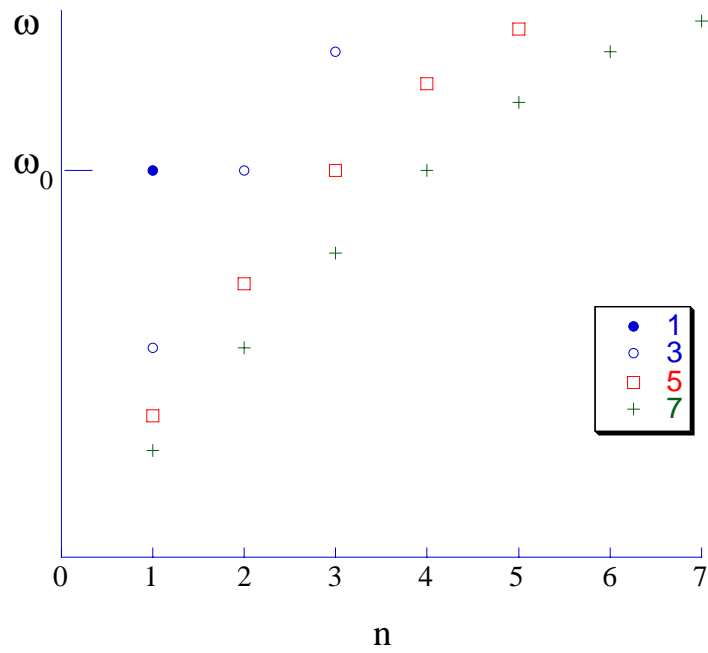
If we only have one mass the mode frequency is $\omega_0 = \sqrt{2k/m}$. For increasing numbers we get

$$1: \quad \omega_0 = \sqrt{2k/m}$$

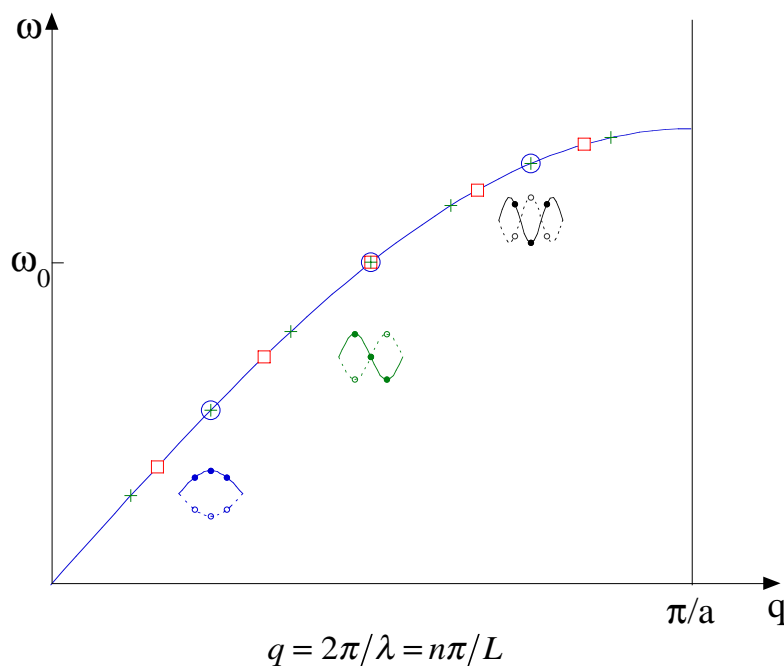
$$3: \quad \omega_0 \times \begin{cases} \sqrt{1 - 1/\sqrt{2}} \\ 1 \\ \sqrt{1 + 1/\sqrt{2}} \end{cases}$$

$$\begin{aligned}
 5: \quad \omega_0 \times & \left\{ \begin{array}{l} \sqrt{1 - \sqrt{3}/\sqrt{2}} \\ \sqrt{1 - 1/2} \\ 1 \\ \sqrt{1 + 1/2} \\ \sqrt{1 + \sqrt{3}/\sqrt{2}} \end{array} \right. \\
 7: \quad \omega_0 \times & \left\{ \begin{array}{l} \sqrt{1 - \sqrt{2 + \sqrt{2}}/2} \\ \sqrt{1 - \sqrt{2}/2} \\ \sqrt{1 - \sqrt{2 - \sqrt{2}}/2} \\ 1 \\ \sqrt{1 + \sqrt{2 - \sqrt{2}}/2} \\ \sqrt{1 + \sqrt{2}/2} \\ \sqrt{1 + \sqrt{2 + \sqrt{2}}/2} \end{array} \right.
 \end{aligned}$$

If we plot these modes they give a rather unorganized impression.



If we on the other hand change the horizontal axis



all points come on one and the same curve, $\omega_q = 2\sqrt{k/m} \sin(qa/2)$. Curves like this are named dispersion curves. In the insets are shown how the masses move when these modes are excited. What is shown is the displacements at the two turning points of the oscillations. The displacements we discuss here are along the springs, so the excitations are longitudinal. The system also has transverse excitations which we do not consider here.

Next we discuss electromagnetic normal modes when two objects are present, surrounded by vacuum. We start with two microscopic objects, two atoms.

Van der Waals interaction between two atoms.

We will now derive the interaction energy and force between two atoms without permanent dipole moments. We will neglect retardation effects; this is allowed except for very large separations between the atoms. We will return to this problem and include retardation effect later; the treatment becomes more complicated but we feel that the picture would not be complete without it.

Johan Diderik van der Waals (1837-1923) graduated on his thesis: *On the continuity of the gaseous and liquid states*, in 1873. He found deviations for

real gases to the ideal gas equation of state. He found empirically

$$\left(p + \frac{a}{V^2}\right)(V - b) = RT ,$$

instead of the ideal equation:

$$pV = RT ,$$

for a mole of gas. The correction constant b is due to the fact that the gas atoms take up a finite fraction of the volume, thus reducing the free volume. The correction a , which is of interest here, is due to the attractive force between the atoms, reducing the pressure exerted on the walls of the container. He was awarded the Nobel Prize in 1910 for this and similar work on the equations of state for gases and fluids.

Van der Waals' force was found on empirical grounds and it was not until 1930 that London gave a realistic explanation for this force. We will go through a similar derivation here. There will be an attractive force between two atoms or molecules even if none of them carry a permanent dipole moment. This is still true if the particles are spherically symmetric. This is a bit counter intuitive. One would imagine that two spherically symmetric, neutral atoms would not interact were they so far apart that their electron wave functions were not overlapping.

We start with two atoms in vacuum. We name them atom 1 and atom 2. Assume that atom 1 is polarized and has a dipole moment \mathbf{p}_1 . A dipole moment gives rise to an electric field which we in chapter 3 found is

$$\mathbf{E} = -\tilde{\phi}\mathbf{p}_1 .$$

The electric field in position of atom 2 is

$$\mathbf{E}_2 = -\tilde{\phi}^{21}\mathbf{p}_1 .$$

Now, this field will polarize atom 2,

$$\mathbf{p}_2 = \tilde{\alpha}_2\mathbf{E}_2 ,$$

where $\tilde{\alpha}_2$ is the polarizability of atom 2. We have here let open the possibility for the atom to be anisotropic by letting the polarizability be a tensor.

This dipole moment gives rise to an electric field, which in the position of atom 1 is

$$\mathbf{E}_1 = -\tilde{\phi}^{12} \mathbf{p}_2.$$

If this field is the cause of the polarization of atom 1 that we started with we have

$$\mathbf{p}_1 = \tilde{\alpha}_1 \mathbf{E}_1.$$

Combining all these equations give

$$\mathbf{p}_1 = \tilde{\alpha}_1 \mathbf{E}_1 = -\tilde{\alpha}_1 \tilde{\phi}^{12} \mathbf{p}_2 = -\tilde{\alpha}_1 \tilde{\phi}^{12} \tilde{\alpha}_2 \mathbf{E}_2 = \tilde{\alpha}_1 \tilde{\phi}^{12} \tilde{\alpha}_2 \tilde{\phi}^{21} \mathbf{p}_1,$$

or

$$\left(\tilde{\mathbb{I}} - \tilde{\alpha}_1 \tilde{\phi}^{12} \tilde{\alpha}_2 \tilde{\phi}^{21} \right) \mathbf{p}_1 = 0.$$

Now we limit the treatment to isotropic atoms and let the polarizabilities be a scalar times the unit tensor, $\tilde{\mathbb{I}}$. We furthermore choose the z -axis to point along the line joining the two atoms, which are r apart. Then

$$\tilde{\phi}^{12} = \tilde{\phi}^{21} = \frac{1}{r^3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

and

$$\tilde{\mathbb{A}} = \begin{pmatrix} 1 - (\alpha_1 \alpha_2) / r^6 & 0 & 0 \\ 0 & 1 - (\alpha_1 \alpha_2) / r^6 & 0 \\ 0 & 0 & 1 - 4(\alpha_1 \alpha_2) / r^6 \end{pmatrix}$$

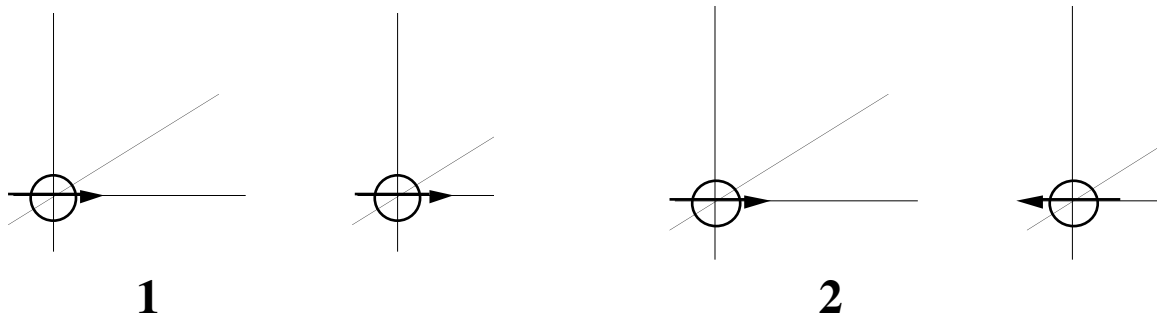
and from which follows that

$$|\tilde{A}| = \left[1 - (\alpha_1\alpha_2)/r^6\right]^2 \left[1 - 4(\alpha_1\alpha_2)/r^6\right].$$

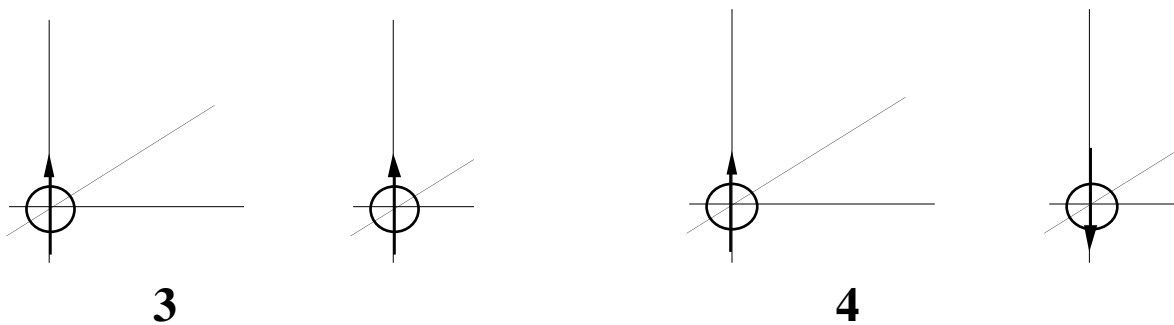
If we have simple enough an expression for the polarizabilities we may now find the mode frequencies by putting the determinant equal to zero. London used a simple approximation for the polarizabilities, the so-called London approximation, which agrees with the Lorentz' classical model that we discussed in chapter 10,

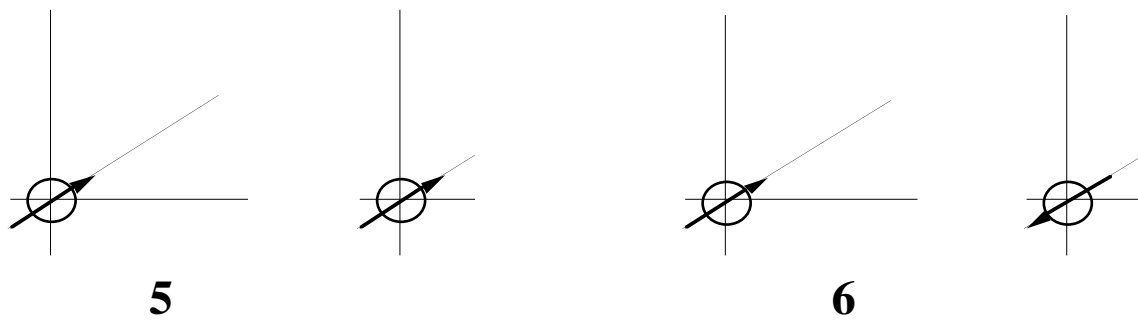
$$\alpha_i(\omega) = \alpha_i(0) / \left[1 - (\omega/\omega_i)^2\right],$$

where ω_i is a characteristic frequency for atom i . The last single factor in the expression for the determinant, above, gives rise to two modes corresponding to oscillating dipoles in the z -direction.

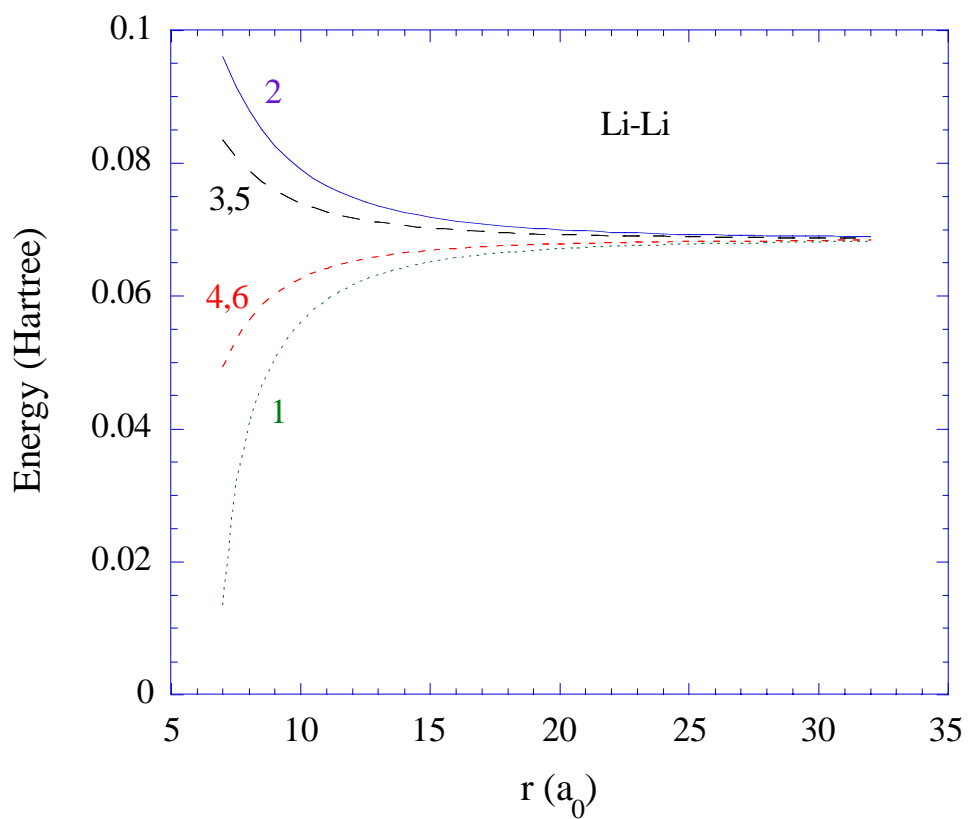


The first squared factor gives four, in pairs degenerate, modes corresponding to dipoles oscillating in the x - and y -directions.





We study an example where both atoms are of the same element, Li.



The interaction energy can be written as

$$E = \sum_{i=1}^6 \left\{ \hbar \omega_i(r) \left[n + \frac{1}{2} \right] - \hbar \omega_i(\infty) \left[n + \frac{1}{2} \right] \right\} = \frac{\hbar}{2} \sum_{i=1}^6 [\omega_i(r) - \omega_i(\infty)],$$

from which the force can be obtained as

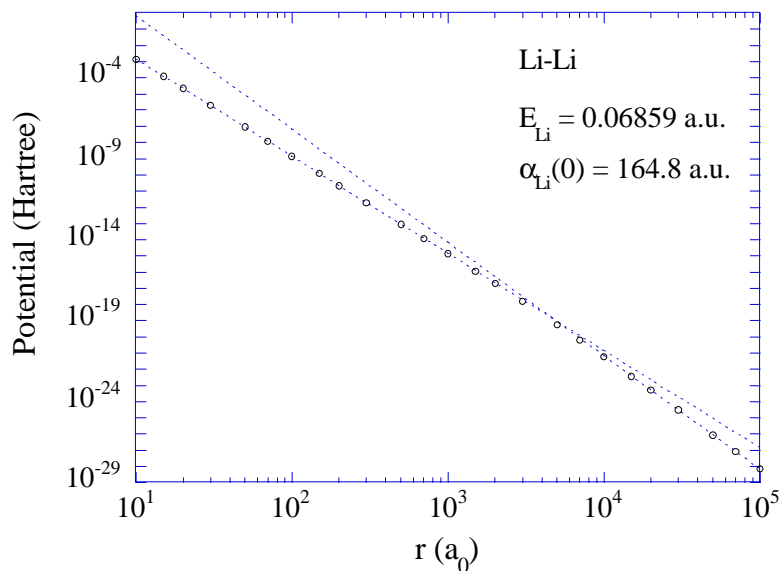
$$F = -\frac{d}{dr}E(r).$$

The modes are so-called mass-less bosons. At zero temperature, in absence of any external stimulation, the occupation number, n , vanishes and zero-point energy, only, contributes. It is the shift of the zero-point energy when the interaction is turned on that contributes to the interaction energy. With our approximations we end up with the following interaction potential for two different atoms:

$$V_{vdW}(r) = -\frac{3\hbar}{2} \frac{\alpha_1(0)\alpha_2(0)}{r^6} \frac{\omega_1\omega_2}{\omega_1 + \omega_2}.$$

If the atoms are the same we have

$$V_{vdW}(r) = -\frac{3\hbar}{4} \frac{\alpha(0)^2}{r^6} \omega_1.$$



In the figure above the dotted straight line with least (largest) slope is the van der Waals (Casimir) asymptote. The circles are from the best ab initio calculation. We see that the best result follows the van der Waals asymptote for smaller separations and the Casimir asymptote for larger separations.

To get these results for the van der Waals interaction we relied on a simple expression for the polarizabilities. We could determine the modes analytically

and just sum their zero-point energies. We can manage also for more complicated polarizabilities with a mathematical treatment based on analytical functions. However, in doing so we loose some of the physical transparency. This method is based on the so called generalized argument principle

We will repeatedly make use of an extension to the so-called *Argument Principle* familiar to most of us from under-graduate mathematical courses on analytical functions. Let us study a region in the complex frequency plane. We have two functions defined in this region; one, $\phi(z)$, is analytical in the whole region; one, $f(z)$, has poles and zeros inside the region. The following relation holds for an integration path around the region:

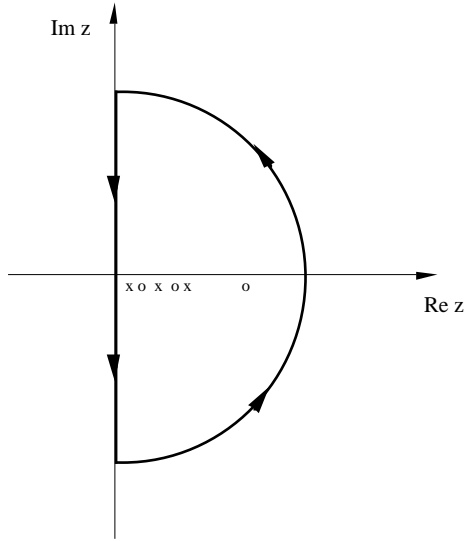
$$\frac{1}{2\pi i} \oint dz \phi(z) \frac{d}{dz} \ln f(z) = \sum \phi(z_o) - \sum \phi(z_\infty),$$

where z_0 and z_∞ are the zeros and poles, respectively, of function $f(z)$. In the Argument Principle the function ϕ is replaced by unity and the right hand side then equals the number of zeros minus the number of poles, for function $f(z)$, inside the integration path.

For the right hand side of the equation above, to produce the interaction energy, we make the choice

$$\phi(z) = \hbar z / 2 ; f(z) = |\tilde{A}|.$$

The contour should include the whole of the positive real frequency axis. The function $f(z)$ is the function in the defining equation for the normal modes of the system. By using this theorem we end up with an integration along a closed contour in the complex frequency plane. In most cases it is fruitful to choose the contour shown in the figure below.



Integration contour in the complex z -plane. Crosses and circles are poles and zeros, respectively, of the function $f(z)$. The radius of the circle is let to go to infinity.

We have the freedom to multiply the function $f(z)$ with an arbitrary constant without changing the zeros and poles. If we choose it carefully we can make the contribution from the curved part of the contour vanish and we are only left with an integration along the imaginary frequency axis.

$$V(r) = -\frac{\hbar}{4\pi} \int_{-\infty}^{\infty} d\omega \omega \frac{d}{d\omega} \ln \left\{ \left[1 - \alpha_1'(\omega) \alpha_2'(\omega) / r^6 \right]^2 \left[1 - 4\alpha_1'(\omega) \alpha_2'(\omega) / r^6 \right] \right\},$$

where the primes on the polarizabilities denote that they are to be calculated on the imaginary frequency axis. Performing a partial integration leads to

$$V(r) = \frac{\hbar}{4\pi} \int_{-\infty}^{\infty} d\omega \ln \left\{ \left[1 - \alpha_1'(\omega) \alpha_2'(\omega) / r^6 \right]^2 \left[1 - 4\alpha_1'(\omega) \alpha_2'(\omega) / r^6 \right] \right\}.$$

For large separation we may expand the logarithm and obtain the van der Waals result

$$V_{vdW}(r) = -\frac{3\hbar}{2\pi} \frac{1}{r^6} \int_{-\infty}^{\infty} d\omega \alpha_1'(\omega) \alpha_2'(\omega) = -\frac{3\hbar}{\pi} \frac{1}{r^6} \int_0^{\infty} d\omega \alpha_1'(\omega) \alpha_2'(\omega).$$

Using the London approximation for the polarizabilities gives

$$V_{vdW}(r) = -\frac{3\hbar}{2} \frac{\alpha_1(0)\alpha_2(0)}{r^6} \frac{\omega_1\omega_2}{\omega_1 + \omega_2}.$$

The results we have obtained here are also valid for spheres and, at large enough distances, for all objects. We see that the interaction potential varies as r^{-6} and the force as r^{-7} .

Retardation effects and the Casimir force

We will now extend the above treatment to be valid for such large distances that the finite speed of light affects the results. To do that we have to make use of the electric field from a time dependent dipole. It is according to the text book in section 9.3

$$\mathbf{E}(\mathbf{r}) = -[\hat{\mathbf{p}} - (\hat{\mathbf{p}} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}] \frac{1}{c^2 r} \ddot{p}\left(t - \frac{r}{c}\right) - [\hat{\mathbf{p}} - 3(\hat{\mathbf{p}} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}] \left[\frac{1}{cr^2} \dot{p}\left(t - \frac{r}{c}\right) + \frac{1}{r^3} p\left(t - \frac{r}{c}\right) \right].$$

It contains contributions from the dipole, its time derivative and its second time derivative; all functions at the retarded time. The electric field from atom 1 at the position of atom 2 is now

$$\mathbf{E}_2(t) = -\tilde{\phi}^{21} [\dot{\mathbf{p}}_1(t - r/c)(r/c) + \mathbf{p}_1(t - r/c)] - \tilde{\vartheta}^{21} \ddot{\mathbf{p}}_1(t - r/c)(r/c)^2,$$

where

$$\tilde{\vartheta}^{21} = \tilde{\vartheta}^{12} = \frac{1}{r^3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

if we let the z -axis point along the line joining the two atoms. The Fourier transformed version is

$$\begin{aligned} \mathbf{E}_2(\omega) &= -\tilde{\phi}^{21} \left[\mathbf{p}_1(\omega) e^{i\omega r/c} (-i\omega r/c) + \mathbf{p}_1(\omega) e^{i\omega r/c} \right] - \tilde{\vartheta}^{21} \mathbf{p}_1(\omega) e^{i\omega r/c} (-i\omega r/c)^2 \\ &= -\mathbf{p}_1(\omega) e^{i\omega r/c} \left[\tilde{\phi}^{21} (1 - i\omega r/c) + \tilde{\vartheta}^{21} (-i\omega r/c)^2 \right]. \end{aligned}$$

So in analogy with the previous section we now arrive at

$$\begin{aligned} \left(\tilde{1} - \tilde{\alpha}_1 e^{i\omega r/c} \left[\tilde{\phi}^{12} (1 - i\omega r/c) + \tilde{\vartheta}^{12} (-i\omega r/c)^2 \right] \tilde{\alpha}_2 e^{i\omega r/c} \right. \\ \left. \times \left[\tilde{\phi}^{21} (1 - i\omega r/c) + \tilde{\vartheta}^{21} (-i\omega r/c)^2 \right] \right) \mathbf{p}_1 = 0. \end{aligned}$$

Assuming isotropic atoms we get

$$\left(\tilde{1} - \alpha_1 \alpha_2 e^{i2\omega r/c} \left[\tilde{\phi} (1 - i\omega r/c) + \tilde{\vartheta} (-i\omega r/c)^2 \right]^2 \right) \mathbf{p}_1 = 0.$$

With our choice of axes the matrix \tilde{A} is diagonal with elements

$$\tilde{A}_{11} = \tilde{A}_{22} = 1 - \alpha_1 \alpha_2 e^{i2\omega r/c} \left[1 - i2(\omega r/c) - 3(\omega r/c)^2 + i2(\omega r/c)^3 + (\omega r/c)^4 \right]$$

and

$$\tilde{A}_{33} = 1 - 4\alpha_1 \alpha_2 e^{i2\omega r/c} \left[1 - i2(\omega r/c) + (\omega r/c)^2 \right].$$

We want the determinant along the imaginary frequency axis. Here it is an even function in ω .

$$\begin{aligned} |\tilde{A}(i\omega)| &= \left[1 - e^{-2|\omega|r/c} \alpha_1'(\omega) \alpha_2'(\omega) \frac{1}{r^6} \left[1 + 2\left(\frac{|\omega|r}{c}\right) + 3\left(\frac{\omega r}{c}\right)^2 + 2\left(\frac{|\omega|r}{c}\right)^3 + \left(\frac{\omega r}{c}\right)^4 \right] \right]^2 \\ &\quad \times \left[1 - 4e^{-2|\omega|r/c} \alpha_1'(\omega) \alpha_2'(\omega) \frac{1}{r^6} \left[1 + 2\left(\frac{|\omega|r}{c}\right) + \left(\frac{\omega r}{c}\right)^2 \right] \right]; \quad \alpha_i'(\omega) = \alpha_i(i\omega) \end{aligned}$$

The interaction potential becomes

$$V(r) = \frac{\hbar}{4\pi} \int_{-\infty}^{\infty} d\omega \ln \left\{ \left[1 - \frac{4e^{-2|\omega|r/c}}{r^6} \alpha_1'(\omega) \alpha_2'(\omega) \left[1 + 2\left(\frac{|\omega|r}{c}\right) + \left(\frac{\omega r}{c}\right)^2 \right] \right] \right. \right. \\ \left. \left. \times \left[1 - \frac{e^{-2|\omega|r/c}}{r^6} \alpha_1'(\omega) \alpha_2'(\omega) \left[1 + 2\left(\frac{|\omega|r}{c}\right) + 3\left(\frac{\omega r}{c}\right)^2 + 2\left(\frac{|\omega|r}{c}\right)^3 + \left(\frac{\omega r}{c}\right)^4 \right] \right] \right]^2 \right\}$$

or

$$V(r) = \frac{\hbar}{2\pi} \int_0^{\infty} d\omega \ln \left\{ \left[1 - \frac{4e^{-2\omega r/c}}{r^6} \alpha_1'(\omega) \alpha_2'(\omega) \left[1 + 2\left(\frac{\omega r}{c}\right) + \left(\frac{\omega r}{c}\right)^2 \right] \right] \right. \right. \\ \left. \left. \times \left[1 - \frac{e^{-2\omega r/c}}{r^6} \alpha_1'(\omega) \alpha_2'(\omega) \left[1 + 2\left(\frac{\omega r}{c}\right) + 3\left(\frac{\omega r}{c}\right)^2 + 2\left(\frac{\omega r}{c}\right)^3 + \left(\frac{\omega r}{c}\right)^4 \right] \right] \right]^2 \right\}$$

For large distances the logarithm may be expanded and only the lowest order term be kept. With large distances we here mean that they are large enough for the interaction to be weak. Then we find:

$$V_{CP}(r) = -\frac{\hbar}{\pi r^6} \int_0^{\infty} d\omega \alpha_1(i\omega) \alpha_2(i\omega) e^{-2\omega r/c} \left[3 + 6(\omega r/c) + 5(\omega r/c)^2 \right. \\ \left. + 2(\omega r/c)^3 + (\omega r/c)^4 \right]$$

This is the Casimir-Polder interaction and it gives the van der Waals result for intermediate separations and the retarded result for large separations. We will now demonstrate that this is the case. Let us first start with the van der Waals limit. Assume that $\omega r/c$ is small compared to unity. The expression in the square brackets reduces to 3 and the exponential prefactor to unity:

$$V_{CP}(r) \underset{\omega r/c \rightarrow 0}{\approx} -\frac{3\hbar}{\pi} \frac{1}{r^6} \int_0^{\infty} d\omega \alpha_1(i\omega) \alpha_2(i\omega)$$

This is the van der Waals result. To find the other limiting result we make the substitution $u = \omega r/c$. Then we have:

$$V_{CP}(r) = -\frac{\hbar c}{\pi r^7} \int_0^{\infty} du \alpha_1\left(i\frac{uc}{r}\right) \alpha_2\left(i\frac{uc}{r}\right) e^{-2u} \left[3 + 6u + 5u^2 + 2u^3 + u^4\right]$$

The exponential factor guarantees that only small u values contribute to the integral. If r is big enough we can replace the polarizabilities with the static ones and move them outside the integral. Then we have:

$$\begin{aligned} V_{CP}(r) &\underset{r \rightarrow \infty}{\approx} -\frac{\hbar c \alpha_1(0) \alpha_2(0)}{\pi r^7} \int_0^{\infty} du e^{-2u} \left[3 + 6u + 5u^2 + 2u^3 + u^4\right] \\ &= -\frac{\hbar c \alpha_1(0) \alpha_2(0)}{\pi r^7} \frac{23}{4} = -\frac{23\hbar c}{4\pi} \left[\alpha_1(0) \alpha_2(0)\right] \frac{1}{r^7} \end{aligned}$$

which is the Casimir result. Thus we see that for intermediate separations the potential goes as r^{-6} and for large separations as r^{-7} . Next we study macroscopic objects.

Electromagnetic normal modes are solutions to the homogeneous versions of Maxwell's equations, i.e., the equations when the sources are absent. The system we are concerned with here consists of two objects surrounded by vacuum. We may divide the modes into three groups: vacuum modes, i.e., modes of the surrounding vacuum; bulk modes, i.e., modes inside the objects far away from the surfaces; surface modes, i.e., modes bound to the surfaces of the objects.

The vacuum modes we have already obtained in chapter 9. They are transverse plane waves characterized by a 3D wave vector, \mathbf{q} , where the \mathbf{E} and \mathbf{B} fields are both mutually perpendicular and perpendicular to \mathbf{q} ; their amplitudes are equal; they are in phase. The dispersion curve is

$$\omega = c\mathbf{q}.$$

The energy and momentum for a mode \mathbf{q} is

$$E = \hbar\omega_{\mathbf{q}}; \quad \mathbf{p}_{\mathbf{q}} = \hbar\mathbf{q}.$$

Also the bulk modes we have already touched upon in chapter 11. There are both longitudinal and transverse types. The longitudinal electric modes are

obtained from the relation

$$\tilde{\varepsilon}(\mathbf{q}, \omega_{\mathbf{q}}) = 0,$$

the longitudinal magnetic modes from

$$\tilde{\mu}(\mathbf{q}, \omega_{\mathbf{q}}) = 0,$$

and the transverse electromagnetic from

$$\tilde{\varepsilon}(\mathbf{q}, \omega_{\mathbf{q}}) \tilde{\mu}(\mathbf{q}, \omega_{\mathbf{q}}) (\omega_{\mathbf{q}}/c)^2 - q^2 = 0.$$

The dispersion relations are found implicitly from the above relations. In most cases we have non-magnetic materials. Then the two type of modes are obtained from the relations

$$\tilde{\varepsilon}(\mathbf{q}, \omega_{\mathbf{q}}) = 0,$$

and

$$\tilde{\varepsilon}(\mathbf{q}, \omega_{\mathbf{q}}) (\omega_{\mathbf{q}}/c)^2 - q^2 = 0,$$

respectively. The bulk modes are important for the stability of the object. The interactions among the particles within the object and the interaction energies can be expressed in terms of these modes. The longitudinal modes are most important here.

The third group of modes, the surface modes, we have not yet discussed. These modes are solutions to Maxwell's equations with the proper boundary conditions; they are localized to the surfaces and interfaces. They play an important role in many situations. One set of effects derive from the fact that these modes contain energy; this means that the modes contribute to the surface energy and surface tension of the objects, which means that they have effect on the stability and shape of the object itself; they also give rise to forces between objects, forces that are of particular importance if the objects are small. The forces are always there but sometimes their effects are masked by the

presence of stronger forces. Another effect is that their presence modifies the optical properties of the objects; with the term optical properties we do not mean to limit ourselves to the visible range of the spectrum. The energy of each mode is sensitive to small changes at the surface like the presence of other atoms. These changes can be detected by optical means or other. Thus, the modes can be utilized, and are so, in sensors like gas sensors. The forces between objects are used in the atomic force microscope and similar instruments.

In small objects the modes on different parts of the surface affect each other and the geometry has to be taken into account when Maxwell's equations are solved. For larger objects the surface can locally be considered flat and one may use the results for planar interfaces. The surface modes at a planar interface are characterized by the 2D wave-vector, \mathbf{k} . We will here only consider planar interfaces. Solving Maxwell's equations on both sides of the interface and using the boundary conditions that the tangential component of the \mathbf{E} - and \mathbf{H} -fields and the normal component of the \mathbf{D} - and \mathbf{B} -fields are continuous one arrives at the following condition for modes

$$\tilde{\epsilon}_1(\omega_{\mathbf{k}})\sqrt{k^2 - \tilde{\epsilon}_2(\omega_{\mathbf{k}})(\omega_{\mathbf{k}}/c)^2} + \tilde{\epsilon}_2(\omega_{\mathbf{k}})\sqrt{k^2 - \tilde{\epsilon}_1(\omega_{\mathbf{k}})(\omega_{\mathbf{k}}/c)^2} = 0.$$

We note that a necessary condition is that the dielectric function has different sign on the two sides of the interface. Neglecting retardation effects, i.e., treating the speed of light as infinite gives

$$\tilde{\epsilon}_1(\omega_{\mathbf{k}}) + \tilde{\epsilon}_2(\omega_{\mathbf{k}}) = 0.$$

Metal-vacuum interface

We are interested in a metal surface in vacuum. In this case we have for small momenta

$$\epsilon_1 \approx 1 - \frac{\omega_{pl}^2}{\omega^2}; \epsilon_2 = 1,$$

which gives for the surface plasmon energy, the energy of the surface mode

obtained neglecting retardation effects,

$$\omega_s = \frac{\omega_{pl}}{\sqrt{2}}.$$

The equation determining the modes when retardation effects are taken into account has the solution

$$\begin{aligned} \omega^2 &= \frac{1}{2} \left[\left(\omega_{pl}^2 + 2c^2 k^2 \right) - \sqrt{\omega_{pl}^4 + 4c^4 k^4} \right] \\ &= \left[\left(\omega_s^2 + c^2 k^2 \right) - \sqrt{\omega_s^4 + c^4 k^4} \right]. \end{aligned}$$

This result gives the surface plasmon dispersion. It is an important result if we want to calculate surface energies of metals. The mode approaches the non-retarded result for large momentum. The deviation from this result only occurs for very small momentum, near the light dispersion curve. We will find later that the retardation effects are important for the force between objects at large separation only. The surface plasmon dispersion is shown in the figure below.

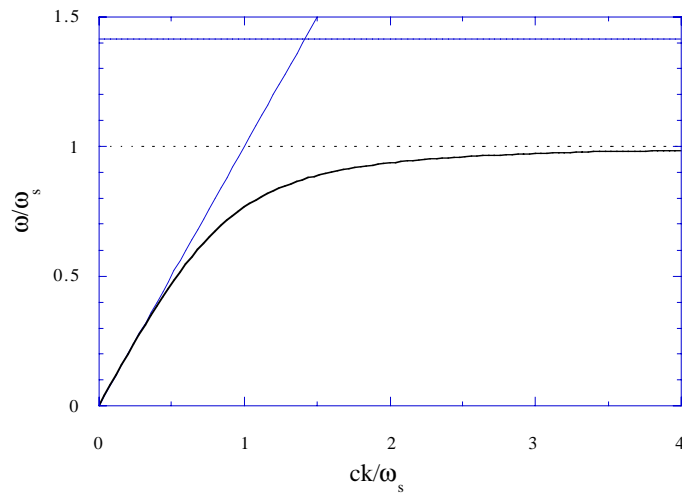


Figure The surface mode for a metal-vacuum interface. The solid horizontal line is the plasmon energy and the dotted is the surface plasmon energy. The diagonal straight line is the light dispersion curve in vacuum and the curved solid curve is the surface plasmon dispersion.

We see that the modes are close to the non-retarded result for large momentum and are pushed below the light dispersion curve for small

momentum. They are only present in the frequency range where the dielectric function of the metal is negative and they are confined to the region to the right of the light dispersion curve for the medium outside the metal surface.

The modes are by some named surface polaritons for general materials, or surface plasmon polaritons in this special case, since they contain a photon component; some reserve the name polariton to the solution near the light dispersion curve and call them surface plasmons for larger momentum; some call them surface plasmons in the whole momentum range.

Semiconductor-vacuum interface

In the present case we have

$$\varepsilon_1 = \varepsilon_\infty \frac{\omega^2 - \omega_L^2}{\omega^2 - \omega_T^2}; \quad \varepsilon_2 = 1.$$

The frequencies ω_L and ω_T are the long-wavelength-limiting frequencies of the longitudinal optical and transverse optical vibration modes of the crystal. The condition for modes becomes

$$\omega_{\mathbf{k}}^2 = \frac{1}{2} \left[\left(\omega_q^2 + \omega_L^2 \right) \pm \sqrt{\left(\omega_q^2 + \omega_L^2 \right)^2 - 4\omega_q^2 \omega_{sph}^2} \right],$$

where

$$\omega_q^2 = \frac{\varepsilon_\infty + 1}{\varepsilon_\infty} (ck)^2$$

$$\omega_L^2 = \frac{\varepsilon_0}{\varepsilon_\infty} \omega_T^2$$

$$\omega_{sph}^2 = \frac{\varepsilon_0 + 1}{\varepsilon_\infty + 1} \omega_T^2.$$

The non-retarded result is ω_{sph} . The result is shown in the figure below

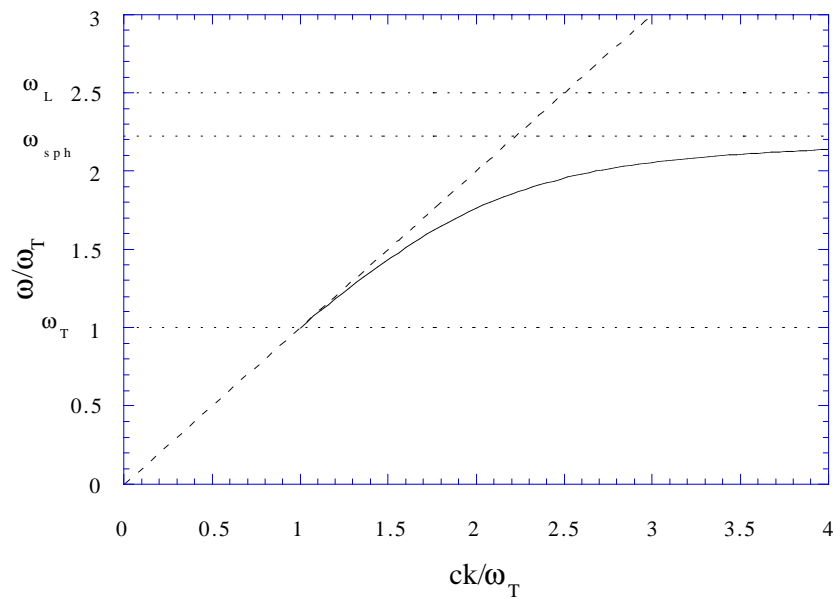


Figure Surface phonon mode in a polar semiconductor.

We find that the surface phonon mode only exists in the frequency range between the transverse and longitudinal phonon modes. It approaches the non-retarded result for high momentum and is pushed down below the light dispersion curve for small momentum. The modes are called surface phonons or surface phonon polaritons.

Force between two halfspaces

To find the modes in the case of two halfspaces, separated a distance d from each other with vacuum between, one first solves Maxwell's equation in the three regions and uses the standard boundary conditions at the two interfaces. We first neglect retardation effects, i.e., we let the speed of light in vacuum be infinite. This is much simpler than doing the full calculation. Maxwell's equations in absence of external charge densities then look like

$$\nabla \cdot \tilde{\mathbf{D}} = 0$$

$$\nabla \cdot \tilde{\mathbf{B}} = 0$$

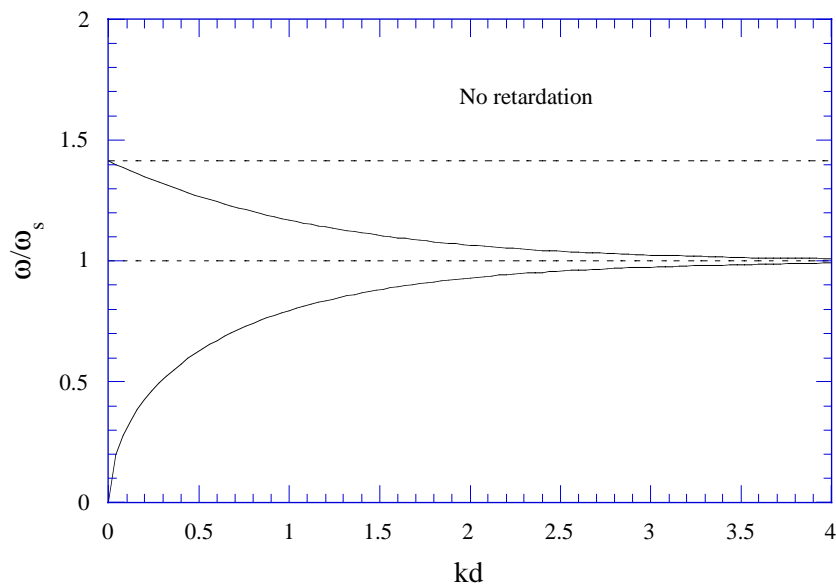
$$\nabla \times \tilde{\mathbf{E}} = 0$$

$$\nabla \times \tilde{\mathbf{H}} = 0.$$

One arrives at the following condition for modes:

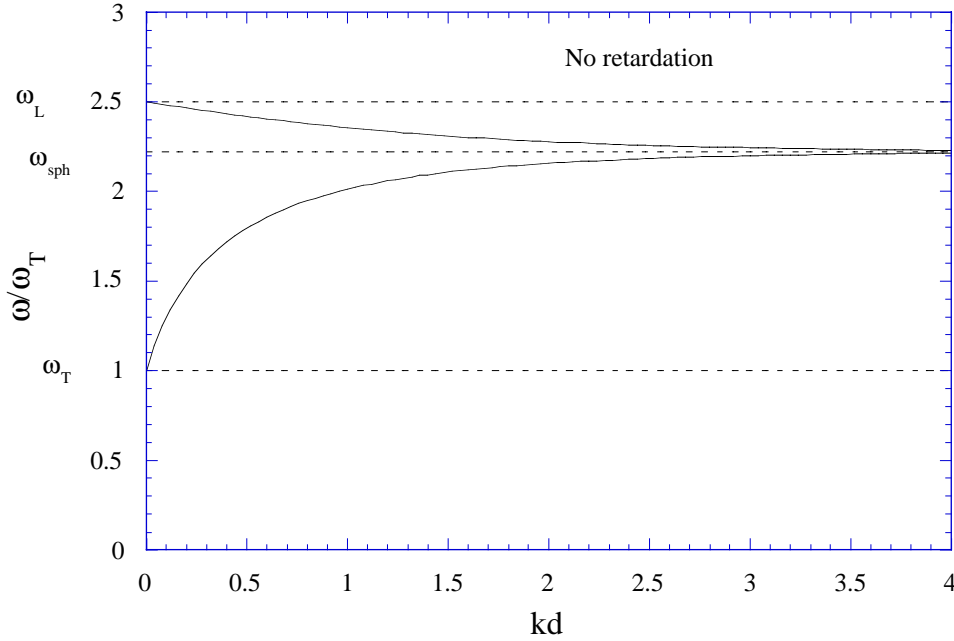
$$[1 + \tilde{\epsilon}(\omega)]^2 - e^{-2kd} [1 - \tilde{\epsilon}(\omega)]^2 = 0,$$

where d is the distance between the two halfspaces of material having dielectric function $\tilde{\epsilon}(\omega)$. For two metal halfspaces the dispersion curves for the two resulting modes are shown in the figure below



These two modes derive from the surface modes on the two surfaces. There are no vacuum modes involved; when retardation is neglected there are no vacuum modes.

For two polar semiconductor halfspaces the dispersion curves are:



The interaction energy per unit area, when we choose our energy reference to be at infinite separation is

$$\begin{aligned}
 V(d) &= \frac{\hbar}{2} \iint \frac{d\omega}{2\pi} \frac{d^2k}{(2\pi)^2} \ln \left\{ 1 - e^{-2kd} \frac{[1 - \varepsilon'(\omega)]^2}{[1 + \varepsilon'(\omega)]^2} \right\} \\
 &= \frac{\hbar}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_0^{\infty} \frac{2\pi k dk}{(2\pi)^2} \ln \left\{ 1 - e^{-2kd} \frac{[1 - \varepsilon'(\omega)]^2}{[1 + \varepsilon'(\omega)]^2} \right\} \\
 &= \frac{\hbar}{4\pi^2} \int_0^{\infty} d\omega \int_0^{\infty} k dk \ln \left\{ 1 - e^{-2kd} \frac{[1 - \varepsilon'(\omega)]^2}{[1 + \varepsilon'(\omega)]^2} \right\} \\
 &= \frac{\hbar}{4\pi^2 d^2} \int_0^{\infty} d\omega \int_0^{\infty} x dx \ln \left\{ 1 - e^{-2x} \frac{[1 - \varepsilon'(\omega)]^2}{[1 + \varepsilon'(\omega)]^2} \right\}.
 \end{aligned}$$

We see that the interaction energy goes as d^{-2} and hence the force as d^{-3} .

Now we include retardation effects. The calculations become much more elaborate. One finds two types of mode, TM (Transverse Magnetic) and TE (Transverse Electric). One arrives at the following conditions for modes:

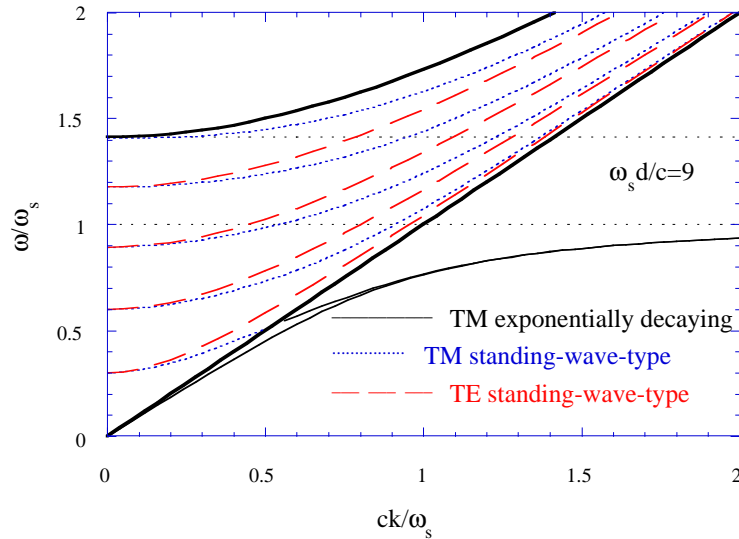
TM:

$$\left[\varepsilon(\omega) + \sqrt{k^2 - \varepsilon(\omega)(\omega/c)^2} / \sqrt{k^2 - (\omega/c)^2} \right]^2 - e^{-2\sqrt{k^2 - (\omega/c)^2}d} \left[\varepsilon(\omega) - \sqrt{k^2 - \varepsilon(\omega)(\omega/c)^2} / \sqrt{k^2 - (\omega/c)^2} \right]^2 = 0,$$

TE:

$$\left[\sqrt{k^2 - \varepsilon(\omega)(\omega/c)^2} + \sqrt{k^2 - (\omega/c)^2} \right]^2 - e^{-2\sqrt{k^2 - (\omega/c)^2}d} \left[\sqrt{k^2 - \varepsilon(\omega)(\omega/c)^2} - \sqrt{k^2 - (\omega/c)^2} \right]^2 = 0.$$

We show the results for simple Drude metals in the figure below.



The two surface modes are modified and one even crosses the light dispersion curve. New standing wave type of modes appear originating from the vacuum modes.

This leads to

$$V(d) = \frac{\hbar}{4\pi^2} \int_0^\infty \int_0^\infty d\omega dk \ln \left\{ 1 - e^{-2\sqrt{k^2 + \left(\frac{\omega}{c}\right)^2} d} \frac{\left[\varepsilon'(\omega) \sqrt{k^2 + \left(\frac{\omega}{c}\right)^2} - \sqrt{k^2 + \varepsilon'(\omega) \left(\frac{\omega}{c}\right)^2} \right]^2}{\left[\varepsilon'(\omega) \sqrt{k^2 + \left(\frac{\omega}{c}\right)^2} + \sqrt{k^2 + \varepsilon'(\omega) \left(\frac{\omega}{c}\right)^2} \right]^2} \right\} \\ + \frac{\hbar}{4\pi^2} \int_0^\infty \int_0^\infty d\omega dk \ln \left\{ 1 - e^{-2\sqrt{k^2 + \left(\frac{\omega}{c}\right)^2} d} \frac{\left[\sqrt{k^2 + \varepsilon'(\omega) \left(\frac{\omega}{c}\right)^2} - \sqrt{k^2 + \left(\frac{\omega}{c}\right)^2} \right]^2}{\left[\sqrt{k^2 + \varepsilon'(\omega) \left(\frac{\omega}{c}\right)^2} + \sqrt{k^2 + \left(\frac{\omega}{c}\right)^2} \right]^2} \right\}.$$

Here it is difficult to use a transformation to find the distance dependence since now both momentum and frequency appear in the exponent. For large separations however we can only have contributions for both k and ω/c small. This is taken care of by the exponential factor in the integrands. This means that the dielectric function may be replaced by its static value:

$$V(d)|_{d \text{ large}} = \frac{\hbar}{4\pi^2} \int_0^\infty \int_0^\infty d\omega dk \ln \left\{ 1 - e^{-2\sqrt{k^2 + \left(\frac{\omega}{c}\right)^2} d} \frac{\left[\varepsilon'(0) \sqrt{k^2 + \left(\frac{\omega}{c}\right)^2} - \sqrt{k^2 + \varepsilon'(0) \left(\frac{\omega}{c}\right)^2} \right]^2}{\left[\varepsilon'(0) \sqrt{k^2 + \left(\frac{\omega}{c}\right)^2} + \sqrt{k^2 + \varepsilon'(0) \left(\frac{\omega}{c}\right)^2} \right]^2} \right\} \\ + \frac{\hbar}{4\pi^2} \int_0^\infty \int_0^\infty d\omega dk \ln \left\{ 1 - e^{-2\sqrt{k^2 + \left(\frac{\omega}{c}\right)^2} d} \frac{\left[\sqrt{k^2 + \varepsilon'(0) \left(\frac{\omega}{c}\right)^2} - \sqrt{k^2 + \left(\frac{\omega}{c}\right)^2} \right]^2}{\left[\sqrt{k^2 + \varepsilon'(0) \left(\frac{\omega}{c}\right)^2} + \sqrt{k^2 + \left(\frac{\omega}{c}\right)^2} \right]^2} \right\}.$$

Now we may make the substitutions $k \rightarrow k/2d$; $\omega \rightarrow \omega/2d$ and find

$$\begin{aligned}
V(d)|_{d \text{ large}} = & \frac{\hbar}{32\pi^2 d^3} \int_0^\infty \int_0^\infty d\omega dk \ln \left\{ 1 - e^{-\sqrt{k^2 + \left(\frac{\omega}{c}\right)^2}} \frac{\left[\varepsilon'(0) \sqrt{k^2 + \left(\frac{\omega}{c}\right)^2} - \sqrt{k^2 + \varepsilon'(0) \left(\frac{\omega}{c}\right)^2} \right]^2}{\left[\varepsilon'(0) \sqrt{k^2 + \left(\frac{\omega}{c}\right)^2} + \sqrt{k^2 + \varepsilon'(0) \left(\frac{\omega}{c}\right)^2} \right]^2} \right\} \\
& + \frac{\hbar}{32\pi^2 d^3} \int_0^\infty \int_0^\infty d\omega dk \ln \left\{ 1 - e^{-\sqrt{k^2 + \left(\frac{\omega}{c}\right)^2}} \frac{\left[\sqrt{k^2 + \varepsilon'(0) \left(\frac{\omega}{c}\right)^2} - \sqrt{k^2 + \left(\frac{\omega}{c}\right)^2} \right]^2}{\left[\sqrt{k^2 + \varepsilon'(0) \left(\frac{\omega}{c}\right)^2} + \sqrt{k^2 + \left(\frac{\omega}{c}\right)^2} \right]^2} \right\}.
\end{aligned}$$

The separation dependence of the interaction energy is now d^{-3} and from this follows that the Casimir force goes as d^{-4} . Thus the force at large separations drops off faster when retardation is included. For a metal the dielectric function diverges for zero frequency. This may be used to find the result:

$$\begin{aligned}
V(d)|_{d \text{ large}} = & \frac{\hbar}{32\pi^2 d^3} \int_0^\infty \int_0^\infty d\omega dk \ln \left\{ 1 - e^{-\sqrt{k^2 + \left(\frac{\omega}{c}\right)^2}} \right\} \\
& + \frac{\hbar}{32\pi^2 d^3} \int_0^\infty \int_0^\infty d\omega dk \ln \left\{ 1 - e^{-\sqrt{k^2 + \left(\frac{\omega}{c}\right)^2}} \right\} \\
= & \frac{\hbar}{16\pi^2 d^3} \int_0^\infty \int_0^\infty d\omega dk \ln \left\{ 1 - e^{-\sqrt{k^2 + \left(\frac{\omega}{c}\right)^2}} \right\} \\
= & \frac{\hbar c}{16\pi^2 d^3} \int_0^\infty \int_0^\infty dx dk \ln \left\{ 1 - e^{-\sqrt{k^2 + x^2}} \right\} \\
= & \frac{\hbar c}{16\pi^2 d^3} \int_0^\infty \int_0^\infty dx dy \ln \left\{ 1 - e^{-\sqrt{y^2 + x^2}} \right\}.
\end{aligned}$$

Treating the x - and y -coordinates as Cartesian coordinates and changing to polar coordinates gives

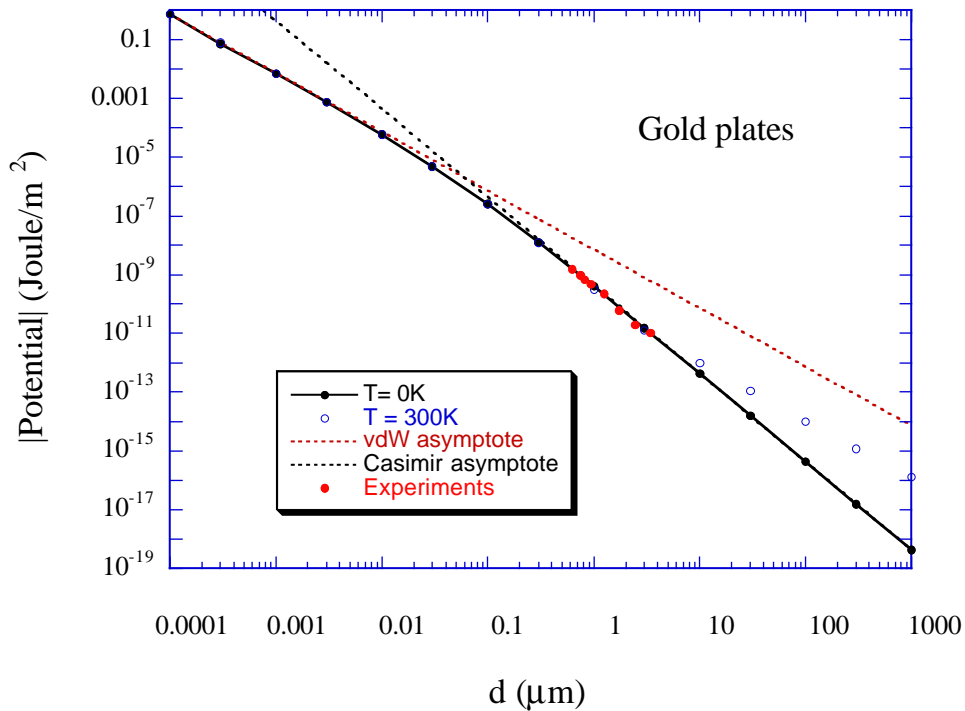
$$\begin{aligned}
 V(d)|_{d \text{ large}} &= \frac{\hbar c}{16\pi^2 d^3} \int_0^\infty dr r \int_0^{\pi/2} d\theta r \cos\theta \ln\{1 - e^{-r}\} \\
 &= \frac{\hbar c}{16\pi^2 d^3} \int_0^\infty dr r^2 \ln\{1 - e^{-r}\} = -\frac{\hbar c \pi^2}{720 d^3},
 \end{aligned}$$

for the energy and

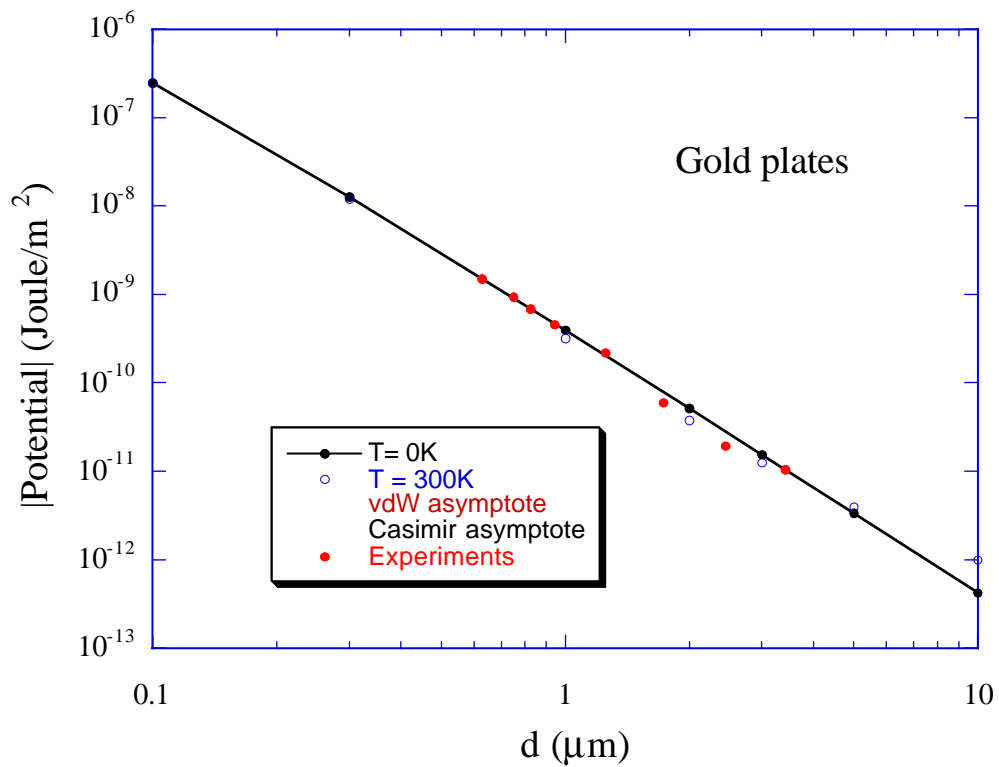
$$F(d)|_{d \text{ large}} = -\frac{\hbar c \pi^2}{240 d^4},$$

for the force. This is what Casimir found for perfect metals, in that case for all separations. For perfect metals there are no van-der-Waals region.

Now, the results of the full calculation for two gold halfspaces, using experimentally obtained dielectric functions, are shown in the figure, below.



In next figure we have expanded the length scale.



The solid curve with filled circles is the result using the equation above. It is valid for zero temperature. The open circles are the corresponding result at room temperature. The red solid circles are the result from modern experiments by Lamoreaux using a torsion pendulum.