DIELECTRIC MEDIA

Microscopic fields ---- Macroscopic fields

Polarization

Microscopic description:

(1) Non-polar molecules get polarized.
The dipole moment tend to point in the direction of the field.
For high frequency fields it may point in the opposite direction.

(2) Polar molecules become partly aligned.
The dipole moment tend to point in the direction of the field

(3) Displacement polarization in polar semiconductors and ionic insulators.

Macroscopic description:

Polarization **P**, electric dipole moment per unit volume.

$\mathbf{P}(\mathbf{r})=\mathbf{p}_0(\mathbf{r})N(\mathbf{r})$

We distinguish between free and polarization or bound charges. The polarization charges or bound charges are related to the polarization according to

$$\rho_b = -div\,\mathbf{P}$$

Gauss' law gives

$$div\,\mathbf{E} = 4\pi\left(\rho_f + \rho_b\right) = 4\pi\rho_f - 4\pi div\,\mathbf{P}$$

or

$$div\,\mathbf{D} = div(\mathbf{E} + 4\pi\mathbf{P}) = 4\pi\rho_f$$

defining Maxwell's *dielectric displacement*

$$\boxed{\mathbf{D} \equiv \mathbf{E} + 4\pi\mathbf{P}}$$

The macroscopic form of Gauss' law in dielectric media:

$$\boxed{div\,\mathbf{D} = 4\pi\rho_f} \qquad \text{1st ME}$$

Thus we have now derived Maxwell's first equation.

Often there is a linear relation between the polarization and the electric field

$$\mathbf{P} = \chi_e\mathbf{E}$$

where χ_e is the electric susceptibility. This means that

$$\mathbf{D} = (1 + 4\pi\chi_e)\mathbf{E}$$

or

$$\boxed{\mathbf{D} = \varepsilon\,\mathbf{E}}$$

where the *dielectric constant* ε is

$$\varepsilon \equiv 1 + 4\pi\chi_e$$

We should bear in mind that this treatment is approximate. It is only valid for linear, isotropic and homogeneous media. For anisotropic materials the constant is a tensor and the **D** and **E** fields are no longer parallel. Furthermore, the constant is really a function of time and position or frequency and momenta. As it stands now the **D** field in a point in space depends on the value of the **E** field in the same point only and at the same time only:

$$\mathbf{D}(\mathbf{r},t) = \varepsilon\,\mathbf{E}(\mathbf{r},t)$$

The correct relation for a linear, isotropic and homogeneous medium is

$$\mathbf{D}(\mathbf{r},t) = \int\int d^3r' dt' \varepsilon(\mathbf{r}-\mathbf{r}', t-t')\mathbf{E}(\mathbf{r}', t')$$

or

$$\mathbf{P}(\mathbf{r},t) = \int\int d^3r' dt' \chi_e(\mathbf{r}-\mathbf{r}', t-t')\mathbf{E}(\mathbf{r}', t')$$

These relations are so-called convolution integrals. It shows that the polarization depends on the electric field in all points in space and in principle at all times. Now, we believe in causality which means that the polarization can not depend on the electric fields at later times, in the future, only on present or past times. This means that

$$\chi_e(\mathbf{r},t) \propto \theta(t)$$

This is common for all so-called time-correlation functions describing the response of the system to a perturbation.

For the simpler relation to hold the susceptibility must be close to a delta function in both its arguments.

The convolution integrals have the very nice property that

$$\mathbf{D}(\mathbf{q},\omega) = \varepsilon(\mathbf{q},\omega)\mathbf{E}(\mathbf{q},\omega)$$

and

$$\mathbf{P}(\mathbf{q},\omega) = \chi_e(\mathbf{q},\omega)\mathbf{E}(\mathbf{q},\omega)$$

i.e., the Fourier transforms have this nice multiplication rule. For this simple relation to hold in "*r* and *t*" space the Fourier transforms of the dielectric function and susceptibility must be constants, i.e., independent of frequency and momentum. This is a good approximation in some limited regions of the ωq-plane. For dielectrics (semiconductors or insulators) it is valid for small momenta and frequencies:

$\omega << E_g / \hbar$

$q << q_{B.Z.}$

When we study electromagnetic fields the neglect of the momentum dependence is usually no big deal, but the frequency dependence is often important.

We now have the 1st and 3rd of Maxwell's equations for static charge distributions and stationary currents. To treat the 2nd and 4th we have to study magnetic fields. This we do next. To get the complete Maxwell's equations we have to include time dependent charge distributions and time-dependent currents.

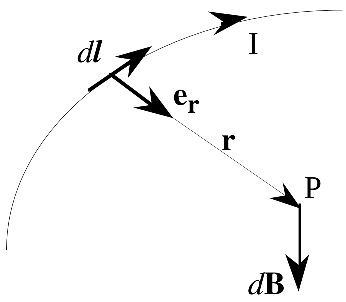
MAGNETIC FIELDS

In the 1820's Ampère and Biot and Savart performed the experiments needed as foundation for the remaining Maxwell's equations. The results are in form of the *Biot-Savart law*:

$$\boxed{\mathbf{B} = \frac{1}{c}\oint \frac{I d\boldsymbol{l} \times \mathbf{e_r}}{r^2}}$$

and the integral form of *Ampère's law*:

$$\boxed{\oint_\Gamma \mathbf{B} \cdot d\boldsymbol{l} = \frac{4\pi}{c} I_{link}}$$



To get the differential form of Ampère's law, Maxwell's fourth equation for static fields, we use Stoke's theorem.

$$\int_S \frac{4\pi}{c}\mathbf{J} \cdot d\mathbf{a} = \underset{\leftarrow}{\left[\frac{4\pi}{c} I = \oint_\Gamma \mathbf{B} \cdot d\boldsymbol{l}\right]}_{\rightarrow} = \int_S \mathbf{curl}\,\mathbf{B} \cdot d\mathbf{a}$$

Because the surface S is arbitrary the integrands must be equal. Thus

$$\boxed{\mathbf{curl}\;\; \mathbf{B} = \frac{4\pi}{c}\mathbf{J}} \qquad \text{4th ME}$$

Now let us use the Biot-Savart law to derive the remaining Maxwell's equation.

$$\mathbf{B}(\mathbf{r}) = \frac{1}{c}\oint_{\Gamma'} \frac{Id\mathbf{r}' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = -\frac{I}{c}\oint_{\Gamma'} d\mathbf{r}' \times \mathbf{grad}_r\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)$$
$$= \mathbf{curl}_r\left(\frac{1}{c}\oint_{\Gamma'} \frac{Id\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\right)$$

In the last step we made use of the relation:

$$\mathbf{curl}(\psi\mathbf{A}) = \psi\mathbf{curl}\mathbf{A} - \mathbf{A} \times \mathbf{grad}\,\psi \quad ; \quad \psi = \frac{1}{|\mathbf{r}-\mathbf{r}'|} \quad ; \quad \mathbf{A} = d\mathbf{r}'$$

In the shorthand notation we may write

$$\mathbf{B} = \mathbf{curl}\left(\frac{1}{c}\oint \frac{I d\boldsymbol{l}}{r}\right)$$

or

$$\boxed{\mathbf{B} = \mathbf{curl}\ \ \mathbf{A}}$$

where we have introduced the *vector potential* **A:**

$$\mathbf{A} = \frac{1}{c}\oint \frac{I d\boldsymbol{l}}{r}$$

Now, since div **curl** = 0 we have

$$\boxed{\text{div}\ \ \mathbf{B} = 0} \qquad \text{2nd ME}$$

which is the remaining Maxwell's equation. It tells that there are no magnetic monopoles. The magnetic field lines have no beginning or end. This equation is sometimes called the *magnetic Gauss' law*.

If we want to calculate the magnetic induction from the (static) source currents we may use one of the following methods:

(1) Calculate the field directly using:

$$\mathbf{B} = \frac{1}{c}\oint \frac{Id\boldsymbol{l} \times \mathbf{e_r}}{r^2}$$

(2) Use the pair of differential equations:

$$\mathbf{curl}\ \ \mathbf{B} = \frac{4\pi}{c}\mathbf{J};$$

$$\text{div}\ \ \mathbf{B} = 0$$

(3) Calculate the vector potential from one of the expressions:

$$\mathbf{A} = \frac{1}{c}\oint \frac{Id\boldsymbol{l}}{r}$$

or

$$\mathbf{A} = \frac{1}{c}\int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

and then use the relation

$$\mathbf{B} = \mathbf{curl}\ \ \mathbf{A}$$

The magnetic force on a moving charge in a magnetic field is

$$\mathbf{F}_{mag} = \frac{q\mathbf{u}}{c} \times \mathbf{B}$$

Note here that this force is perpendicular to the velocity of the charge and does not do any work; the speed is not affected; the direction of motion is affected. There will be a torque though on a current loop.

In presence of both electric and magnetic fields we have

$$\boxed{\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B}\right)}$$

This is the so-called *Lorentz force*.

This expression for the force is <u>valid also for time-varying fields.</u>

For continuous charge and current distributions the force can be written as

$$d\mathbf{F} = \rho_l dl\mathbf{E} + \frac{Id\mathbf{l}}{c} \times \mathbf{B}; \qquad \text{1D}$$

$$d\mathbf{F} = \rho_s da\mathbf{E} + \frac{\mathbf{K}da}{c} \times \mathbf{B}, \qquad \text{2D}$$

$$d\mathbf{F} = \rho dv\mathbf{E} + \frac{\mathbf{J}dv}{c} \times \mathbf{B}; \qquad \text{3D}$$

We treat the current as a macroscopic concept and take away the granular, statistical complications of discrete conduction electrons.

However, in doing so we sometimes miss interesting effects. If we have two parallel, non-charged wires carrying steady state currents we would expect to have no electric fields present and hence no forces from the first term in our force expressions. Now, the granular charge distributions have effects. There will be a so-called *current drag effect* caused by the scattering of the electrons in a wire against the electrons in the other wire. The electric fields or scalar potentials from the individual electrons in a wire reach the other wire and causes the scattering. This is however a small effect. It has been experimentally verified in the last decade.

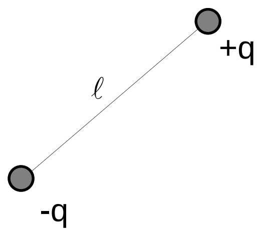
Another effect is the following: Let us have two charge neutral objects close to each other. The fluctuations in the electromagnetic fields from one of the objects due to the individual electrons affects the movement of the electrons in the other object giving rise to collective movements. This reduces the energy and causes an attractive force between the objects; the *van der Waals* and *Casimir* forces. Also within a single object there will be collective movements of the electrons causing a reduction of the electromagnetic energy; this is the so-called *correlation energy*.

MAGNETIC MATERIALS

Magnetization

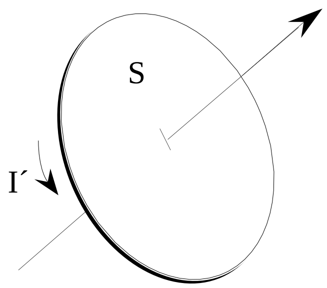
Microscopic description:

Electric dipole moment



$$\mathbf{p} = \lim_{\substack{\ell \to 0 \\ q\ell \text{ constant}}} q\ell \mathbf{e}$$

Magnetic dipole moment



$$\mathbf{m} = \lim_{\substack{S \to 0 \\ I'S \text{ constant}}} \frac{I'}{c}\mathbf{S}$$

(1) Atoms or molecules with *no intrinsic magnetic dipole moment* are distorted in a magnetic field and acquire an induced dipole moment which is typically aligned *antiparallel* with the applied field. This is different from the induction of electric dipoles in an electric field where the dipoles were parallel to the field. It is a result of the Lorentz force. It is also a bit counter intuitive since the energy of electric and magnetic dipoles in electric and magnetic fields, respectively are:

$$U_e = -\mathbf{p} \cdot \mathbf{E} \quad ;$$
$$U_m = -\mathbf{m} \cdot \mathbf{B}$$

and the torques are

$$\tau_e = \mathbf{p} \times \mathbf{E} \quad ;$$
$$\tau_m = \mathbf{m} \times \mathbf{B}$$

The expressions are quite symmetric.
The energy is minimum if the magnetic dipole moment is aligned parallel with the magnetic field.

(2) Atoms or molecules with permanent magnetic dipole moments are preferentially aligned parallel with the applied field. This is fully in line with the equations above.

Macroscopic description:

The magnetization **M** is the magnetic dipole moment per unit volume:

$$\mathbf{M}(\mathbf{r}) = N(\mathbf{r})\mathbf{m}_0(\mathbf{r})$$

The bound magnetization currents are

$$\mathbf{J}_b = c\,\mathbf{curl}\ \mathbf{M}$$

and since

$$\mathbf{curl\ B} = \frac{4\pi}{c}\left(\mathbf{J}_f + \mathbf{J}_b\right)$$

we get

$$\mathbf{curl\ B} = \frac{4\pi}{c}\left(\mathbf{J}_f + c\,\mathbf{curl\ M}\right)$$

or

$$\mathbf{curl}\ (\mathbf{B} - 4\pi\mathbf{M}) = \frac{4\pi}{c}\mathbf{J}_f$$

We define

$$\mathbf{H} \equiv \mathbf{B} - 4\pi\mathbf{M}$$

and

$$\boxed{\mathbf{curl\ H} = \frac{4\pi}{c}\mathbf{J}_f}$$

The macroscopic version of Ampere's law in presence of magnetic media and stationary current densities and static charge densities.

Note the minus sign in

$$\mathbf{H} \equiv \mathbf{B} - 4\pi\mathbf{M}$$

while we have a plus sign in

$$\mathbf{D} \equiv \mathbf{E} + 4\pi\mathbf{P}$$

In para- and dia-magnetic materials **M** is to a good approximation proportional to the magnetic field **H:**

$$\mathbf{M} = \chi_m \mathbf{H}$$

where χ_m is the *magnetic susceptibility*. Note that the relation is between the magnetization and the magnetic field, the **H** field, not with the fundamental **B** field, the magnetic induction. The corresponding relation in dielectric media is between the polarization and electric field.

Thus we have

$$\boxed{\mathbf{B} = \left(1 + 4\pi\chi_m\right)\mathbf{H} \equiv \mu\mathbf{H}}$$

where μ is the *permeability* of the medium.

To be more strict the relation between the magnetization and the magnetic field is on the form:

$$\mathbf{M}(\mathbf{r},t) = \int\int d^3r' dt' \chi_m\left(\mathbf{r} - \mathbf{r}', t - t'\right)\mathbf{H}\left(\mathbf{r}', t'\right)$$

In anisotropic media, χ_m and μ are tensors.

BOUNDARY CONDITIONS

We will now use the result we have found for Maxwell's macroscopic field equations in the case of static charge distributions and stationary current distributions to find the boundary conditions for the fields at boundaries between regions of different materials:

$$\nabla \cdot \mathbf{D} = 4\pi\rho_f \qquad \textit{Gauss' law}$$

$$\nabla \times \mathbf{E} = 0 \qquad \textit{Faraday's law}$$

$$(\textit{Conservative nature of electrostatic forces})$$

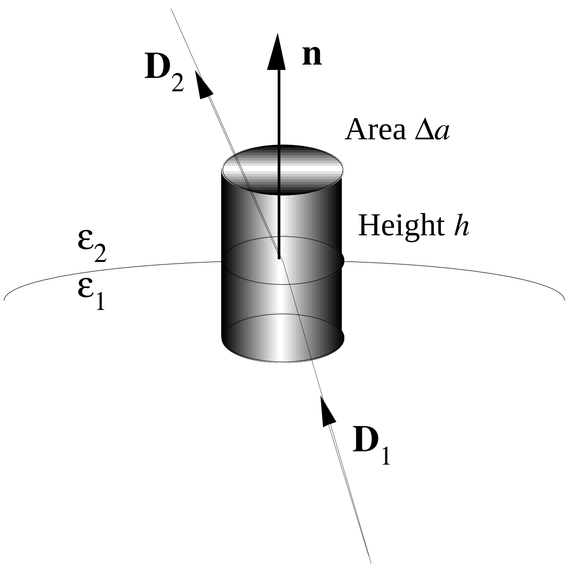
$$\nabla \cdot \mathbf{B} = 0 \qquad \textit{Magnetic Gauss' law}$$

$$(\textit{Absence of free magnetic poles})$$

$$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J}_f \qquad \textit{Ampere's law}$$

We start with the **D**-fields and Gauss' law.

$$4\pi q_f \underset{\leftarrow}{=} \left[\int_V 4\pi\rho_f = \int_V \nabla \cdot \mathbf{D}\right] \underset{\rightarrow}{=} \oint_S \mathbf{D} \cdot \mathbf{n}\, dS$$



Let the height h be very small so that the area of the curved surface of the cylinder is negligible compared to those of the flat surfaces.

This gives for a small "pill box"

$$4\pi\left(\rho_s\right)_f \Delta a = \left[\mathbf{D}_2 \cdot \mathbf{n} - \mathbf{D}_1 \cdot \mathbf{n}\right]\Delta a$$

or

$$\boxed{\left(\mathbf{D}_2 - \mathbf{D}_1\right)\cdot \mathbf{n} = 4\pi\left(\rho_s\right)_f}$$

In absence of free charges the *normal* component of the **D**-vector is *continuous* at an interface. Using one of the constitutive relations we find for the **E**-fields:

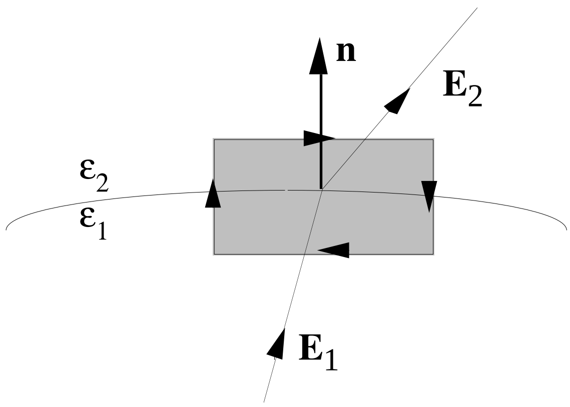
$$\left(\varepsilon_2 \mathbf{E}_2 - \varepsilon_1 \mathbf{E}_1\right)\cdot \mathbf{n} = 4\pi\left(\rho_s\right)_f$$

The normal component of the **E**-field is *not* continuous across an interface even in absence of free charges.

We may also write the following relation

$$\left(\mathbf{E}_2 - \mathbf{E}_1\right)\cdot \mathbf{n} = 4\pi\left(\rho_s\right)_{f+b} = 4\pi\rho_s$$

Next we consider the electric field and make use of Faraday's law.



$$\oint_{\Gamma} \mathbf{E} \cdot d\boldsymbol{l} = \int_{s} (\nabla \times \mathbf{E}) \cdot \mathbf{n}_0 da = 0$$

$$(\mathbf{E}_2 - \mathbf{E}_1) \cdot (\mathbf{n}_0 \times \mathbf{n}) dl = -[(\mathbf{E}_2 - \mathbf{E}_1) \times \mathbf{n}] \cdot \mathbf{n}_0 dl = 0$$

or

$$\boxed{(\mathbf{E}_2 - \mathbf{E}_1) \times \mathbf{n} = 0}$$

The *tangential* component of the **E**-field is *continuous*.

For the **D**-field we have

$$\left(\tfrac{1}{\varepsilon_2} \mathbf{D}_2 - \tfrac{1}{\varepsilon_1} \mathbf{D}_1\right) \times \mathbf{n} = 0$$

The tangential component of the **D**-field is *not* continuous across an interface.

Now we move on to the **B**-field and use the magnetic Gauss' law. The result is obtained in complete analogy with that for the **D**-field:

$$\boxed{(\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{n} = 0}$$

The *normal* component of the **B**-field is continuous across an interface. For the **H**-field we have

$$(\mu_2 \mathbf{H}_2 - \mu_1 \mathbf{H}_1) \cdot \mathbf{n} = 0$$

i.e., the *normal* component of the **H**-field is *not* continuous across an interface.

Finally we treat the **H**-field and the derivation is completely analogous to that for the **E**-field but now we have a surface current density linking the Stokesian rectangle at the interface.

$$\boxed{(\mathbf{H}_2 - \mathbf{H}_1) \times \mathbf{n} = -\frac{4\pi}{c}\mathbf{K}_f}$$

The *tangential* component of the **H**-field is *continuous* across the interface only if there is no free surface current at the interface. Using one constitutive relation gives

$$\left(\frac{\mathbf{B}_2}{\mu_2} - \frac{\mathbf{B}_1}{\mu_1}\right) \times \mathbf{n} = -\frac{4\pi}{c}\mathbf{K}_f$$

which shows that even in absence of free current densities at the interface the *tangential* component of the **B**-field is *not continuous* across the interface.

This last relation may also be written as:

$$(\mathbf{B}_2 - \mathbf{B}_1) \times \mathbf{n} = -\frac{4\pi}{c}(\mathbf{K}_f + \mathbf{K}_b) = -\frac{4\pi}{c}\mathbf{K}$$

We can summarize the boundary conditions in the following way:

In absence of free surface charge densities :

E: Tangential component continuous
D: Normal component continuous
B: Normal component continuous
H: Tangential component continuous

These relations also hold for time dependent fields. The extra terms in Faraday's and Ampere's laws, the time derivatives of the **B** and **D**-fields will give vanishing flux through the Stokesian rectangle when its area goes to zero.