# ELECTRIC FIELD OF A POINT CHARGE IN UNIFORM MOTION

We will now rederive some of the results we have obtained earlier now using the transformation properties of the electromagnetic fields. The first problem is that of the electric field from a charged particle in uniform motion.

We let our charged particle be at rest at the origin in our unprimed reference system. Thus we have

$$\mathbf{E} = q\frac{\mathbf{r}}{r^3}$$

This is the only field we have in the unprimed system.
Let us now for simplicity find the electric field in the primed system at t' = 0 when the charge is at the origin also in the primed inertial frame. We need the following transformations:

$$\mathbf{E}'_{\perp} = \gamma\mathbf{E}_{\perp}; \quad \mathbf{E}'_{//} = \mathbf{E}_{//}$$

$$\left.\begin{aligned} \mathbf{x} &= \mathbf{x}' + \mathbf{v}\left[\frac{\mathbf{x}'\cdot\mathbf{v}}{v^2}(\gamma - 1) - \gamma t'\right] \\ t &= \gamma\left[t' + \frac{\mathbf{x}'\cdot\mathbf{v}}{c^2}\right] \end{aligned}\right\} \rightarrow \mathbf{x}_{//} = \gamma\mathbf{x}'_{//} \quad ; \quad \mathbf{x}_{\perp} = \mathbf{x}'_{\perp}$$

We get

$$
\begin{aligned}
\mathbf{E}' &= \mathbf{E}'_{\perp} + \mathbf{E}'_{//} = \gamma\mathbf{E}_{\perp} + \mathbf{E}_{//} \\
&= \frac{q}{r^3}\left(\gamma\mathbf{r}_{\perp} + \mathbf{r}_{//}\right) = \frac{q}{r^3}\left(\gamma\mathbf{r}'_{\perp} + \gamma\mathbf{r}'_{//}\right) \\
&= \frac{\gamma q\mathbf{r}'}{r^3} = \frac{\gamma q\mathbf{r}'}{\left(r_{\perp}^2 + r_{//}^2\right)^{3/2}} = \frac{\gamma q\mathbf{r}'}{\left(r'^2_{\perp} + \gamma^2 r'^2_{//}\right)^{3/2}} \\
&= \frac{q\mathbf{r}'}{\gamma^2\left(\gamma^{-2} r'^2_{\perp} + r'^2_{//}\right)^{3/2}} = \frac{\left(1-\beta^2\right)q\mathbf{r}'}{\left[\left(1-\beta^2\right)r'^2_{\perp} + r'^2_{//}\right]^{3/2}} \\
&= \frac{\left(1-\beta^2\right)q\mathbf{r}'}{\left[\left(1-\beta^2\right)r'^2\sin^2\theta + r'^2\cos^2\theta\right]^{3/2}} \\
&= q\,\frac{\left(1-\beta^2\right)\mathbf{r}'}{r'^3\left(1-\beta^2\sin^2\theta\right)^{3/2}}
\end{aligned}
$$

The same result as we obtained earlier.

## RADIATION BY AN ACCELERATED CHARGE

The Larmor formula

$$P = \frac{2e^2 a^2}{3c^3}$$

is only valid for an accelerated charge if it moves with non-relativistic speeds. It is exact in the reference frame that is instantaneously at rest with respect to the charge.

The Lorentz transformation is only valid between inertial frames, not accelerating ones. One may still use it however if one is careful. If we have an accelerated particle we can at each instant of time find an inertial frame that has the same velocity as the particle. The exact expression for the radiation in any reference frame $K$ may be obtained by calculating the radiation in the instantaneous rest frame $K'$ according to the Larmor formula and then transforming the result from $K'$ to $K$ by means of the standard relativistic equations.

One can with the help of the four-vector acceleration find the Lorentz transformation of the acceleration. It is

$$a'^2 = \frac{1}{\left(1-\beta^2\right)^2}\left[\dot{\mathbf{u}}\cdot\dot{\mathbf{u}} + \frac{(\mathbf{u}\cdot\dot{\mathbf{u}})^2}{c^2\left(1-\beta^2\right)}\right]$$

Since $W$ and $t$ are each proportional to the fourth component of a four-vector they transform in the same way. Therefore the ratio $dW/dt$ is Lorentz invariant. Thus,

$$P = -\frac{dW}{dt} = -\frac{dW'}{dt'} = P' = \frac{2e^2 a'^2}{3c^3}$$

Hence, the radiated power in $K$ is

$$P = \frac{2e^2}{3c^3} \frac{1}{\left(1-\beta^2\right)^2} \left[ \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} + \frac{(\mathbf{u} \cdot \dot{\mathbf{u}})^2}{c^2\left(1-\beta^2\right)} \right]$$

## RADIATION FROM A CHARGED PARTICLE WITH COLLINEAR VELOCITY AND ACCELERATION

Now we get

$$P = \frac{2e^2}{3c^3\left(1-\beta^2\right)^2} \left[ \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} + \frac{(\mathbf{u} \cdot \dot{\mathbf{u}})^2}{c^2\left(1-\beta^2\right)} \right]$$

$$\downarrow$$

$$P = \frac{2e^2}{3c^3\left(1-\beta^2\right)^2} \left[ a^2 + \frac{u^2 a^2}{c^2\left(1-\beta^2\right)} \right]$$

$$= \frac{2e^2 a^2}{3c^3} \frac{1}{\left(1-\beta^2\right)^3} = \frac{2e^2 a^2}{3c^3} \gamma^6$$

which agrees with what we obtained earlier.

## RADIATION FROM A CHARGED PARTICLE CONFINED TO A CIRCULAR ORBIT

Now the velocity and acceleration are perpendicular, $\mathbf{u}\cdot\dot{\mathbf{u}} = 0$:

$$P = \frac{2e^2}{3c^3\left(1-\beta^2\right)^2}\left[\dot{\mathbf{u}}\cdot\dot{\mathbf{u}} + \frac{(\mathbf{u}\cdot\dot{\mathbf{u}})^2}{c^2\left(1-\beta^2\right)}\right]$$

$$\downarrow$$

$$P = \frac{2e^2a^2}{3c^3\left(1-\beta^2\right)^2} = \frac{2e^2a^2}{3c^3}\gamma^4$$

which is also in agreement with earlier results.

# LORENTZ INVARIANTS

$$ds = \sqrt{dx_\mu dx_\mu} = \sqrt{dx_i dx_i - c^2 dt^2}$$

$$d\tau = \frac{i}{c}\sqrt{dx_\mu dx_\mu} = \sqrt{dt^2 - \frac{1}{c^2} dx_i dx_i} = dt\sqrt{1 - u^2 \;\; c^2}$$

$$\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

$$\mathbf{E} \cdot \mathbf{B} = \mathbf{E}' \cdot \mathbf{B}'$$

$$\mathbf{E}^2 - \mathbf{B}^2 = \mathbf{E}'^2 - \mathbf{B}'^2$$

$$S^2 - c^2 \mathcal{E}^2$$

$$P = -\frac{dW}{dt}$$

## FOUR-VECTORS

To distinguish four-vectors from ordinary three-vectors we add four points above the former. This is not a standard notation.

$$\ddot{\ddot{\mathbf{X}}} = (\mathbf{x}, ict), \quad d\ddot{\ddot{\mathbf{X}}} = (d\mathbf{x}, icdt)$$

$$\ddot{\ddot{\mathbf{U}}} = \frac{d\ddot{\ddot{\mathbf{X}}}}{d\tau} = \left(\frac{d\mathbf{x}}{d\tau}, ic\frac{dt}{d\tau}\right) = \left(\frac{\mathbf{u}}{\sqrt{1-u^2/c^2}}, \frac{ic}{\sqrt{1-u^2/c^2}}\right)$$

$$\ddot{\ddot{\mathbf{P}}} = \left(\frac{m_0\mathbf{u}}{\sqrt{1-u^2/c^2}}, \frac{im_0c}{\sqrt{1-u^2/c^2}}\right) = \left(\mathbf{p}, i\frac{W}{c}\right)$$

$$\mathbf{p} = m\mathbf{u}; \quad m \equiv m_0 \Big/ \sqrt{1-u^2/c^2} \;;$$

$$\mathbf{F} = \frac{d}{dt}\mathbf{p} = \frac{d}{dt}\left(m_0\mathbf{u} \Big/ \sqrt{1-u^2/c^2}\right)$$

$$\ddot{\ddot{\mathbf{J}}} = (\mathbf{J}, ic\rho) = \rho_0\ddot{\ddot{\mathbf{U}}}$$

$$\ddot{\ddot{\mathbf{A}}} = (\mathbf{A}, i\Phi)$$ in Lorentz gauge

$$\ddot{\ddot{\mathbf{D}}} \equiv d\ddot{\ddot{\mathbf{U}}} \big/ d\tau$$ four vector acceleration

$$\ddot{\ddot{\mathbf{K}}} = \frac{1}{c}\{\mathbf{F}\} \cdot \ddot{\ddot{\mathbf{J}}}$$ Lorentz force density

$$\ddot{\ddot{\mathbf{K}}} = (\mathbf{K}, K_4) \;; \quad \mathbf{K} = \rho\left(\mathbf{E} + \frac{1}{c}\mathbf{u} \times \mathbf{B}\right) \;; \quad K_4 = \frac{i}{c}\mathbf{E} \cdot \mathbf{J}$$