ANTENNAS

We have shown the general expressions for the fields from macroscopic distributions of charge and current:

Generalized Coulomb-Faraday law

$$\mathbf{E} = \int_V \left(\frac{[\rho]\mathbf{e}_R}{R^2} + \frac{[\partial\rho/\partial t]\mathbf{e}_R}{cR} - \frac{[\partial \mathbf{J}/\partial t]}{c^2 R} \right) dv'$$

Generalized Biot-Savart law

$$\mathbf{B}(\mathbf{r},t) = \int_V \left(\frac{[\mathbf{J}]\times \mathbf{e}_R}{cR^2} + \frac{[\partial \mathbf{J}/\partial t]\times \mathbf{e}_R}{c^2 R} \right) dv'$$

We also showed the fields from individual charged particles:

$$\mathbf{E} = e\left[\frac{(\mathbf{n}-\boldsymbol{\beta})\left(1-\beta^2\right)}{K^3 R^2} + \frac{\mathbf{n}\times\left((\mathbf{n}-\boldsymbol{\beta})\times \mathbf{a}\right)}{c^2 K^3 R} \right]$$

and

$$\mathbf{B} = [\mathbf{n}]\times \mathbf{E}$$

These expressions can always be used but sometimes one can use simplifications. One such simplification is using multipole expansions.

RADIATION BY MULTIPOLE MOMENTS

Multipole expansions may be used when the sources are limited to a relative small region in space. Then in general there will be relatively strong fields in the neighborhood of the sources that are emitted and reabsorbed by the sources. These are not giving any contribution to the real radiation. To get real radiation one has to have accelerated charges and or time dependent currents.

For large distances from the sources the non-radiative fields have dropped off and the radiation fields, if there are any, dominate.

The multipole expansions can be used in two cases.

a) If the particle motions are slowly varying the near-field non-radiative fields are dominating. These can be obtained to a good approximation by completely neglecting retardation effects and using the same multi-pole expansions as we did for static fields earlier. The multipoles are now time dependent.

b) When the charge accelerations and current variations are large it may be reasonable to neglect the near fields and concentrate on the radiation fields. Here we have some conditions that have to be fulfilled.

i) The retardation within the source must be negligible. This means that the finite extension of the source, d, should be smaller than the wave-length, $d \ll \lambda$.
This is equivalent to assuming $u \ll c$, i.e., non-relativistic particle velocities.

ii) The field point should be much more distant from the source than the extension of the source: $d \ll r$. When we want to study the radiation fields we also have that the distance is much larger than the wavelength, $\lambda \ll r$.
So in the radiation limit we have the conditions: $d \ll \lambda \ll r$.
(In the quasi-static case we have for the near field limit: $d \ll r \ll \lambda$).

ELECTRIC DIPOLE RADIATION

In this limit we can use the second term of the electric field from a single charged particle and sum the contributions from all charges, since **n** can be approximated by the unit vector in the **r** direction and R can be approximated by r. Since we have non-relativistic particle velocities $\boldsymbol{\beta}$ is furthermore negligible.

$$\boxed{\mathbf{E} = e\left[\frac{(\mathbf{n}-\boldsymbol{\beta})\left(1-\beta^2\right)}{K^3R^2} + \frac{\mathbf{n}\times\left((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a}\right)}{c^2K^3R}\right]}$$

Thus we have

$$\begin{aligned}\mathbf{E}_{rad} &= \frac{1}{c^2r}\mathbf{n}\times\left(\mathbf{n}\times\sum_\alpha\left[q_\alpha\mathbf{a}_\alpha\right]\right)\\ &= \frac{1}{c^2r}\mathbf{n}\times\left(\mathbf{n}\times\sum_\alpha\left[q_\alpha\ddot{\mathbf{r}}_\alpha\right]\right)\\ &= \frac{1}{c^2r}\mathbf{n}\times\left(\mathbf{n}\times[\ddot{\mathbf{p}}]\right)\end{aligned}$$

where the summation runs over the particles.

Now from the relation for the magnetic field from a charged particle:

$$\boxed{\mathbf{B} = [\mathbf{n}]\times\mathbf{E}}$$

we get

$$\mathbf{B}_{rad} = \mathbf{n}\times\mathbf{E}_{rad} = -\frac{1}{c^2r}\mathbf{n}\times[\ddot{\mathbf{p}}]$$

Choosing polar coordinates relative the axis of the dipole we get

$$\mathbf{E}_{rad} = \frac{[\ddot{p}]}{c^2 r}\sin\theta\mathbf{e}_\theta$$

$$\mathbf{B}_{rad} = \frac{[\ddot{p}]}{c^2 r}\sin\theta\mathbf{e}_\varphi$$

The Larmor formulas can be directly generalized:

$$\boxed{\frac{dP}{d\Omega} = \frac{e^2 a^2}{4\pi c^3}\sin^2\theta \rightarrow \frac{\left[\ddot{p}^2\right]}{4\pi c^3}\sin^2\theta}$$

$$\boxed{P = \frac{2e^2 a^2}{3c^3} \rightarrow \frac{2\left[\ddot{p}^2\right]}{3c^3}}$$

COMPLETE FIELDS OF A TIME DEPENDENT ELECTRIC DIPOLE

It is possible to derive the complete fields from a linear dipole in the limit:

$d << r$.

These are

$$\boxed{\mathbf{E}(\mathbf{r},t) = \left(\frac{2[p]}{r^3} + \frac{2[\dot{p}]}{cr^2}\right)\cos\theta\mathbf{e}_r + \left(\frac{[p]}{r^3} + \frac{[\dot{p}]}{cr^2} + \frac{[\ddot{p}]}{c^2 r}\right)\sin\theta\mathbf{e}_\theta}$$

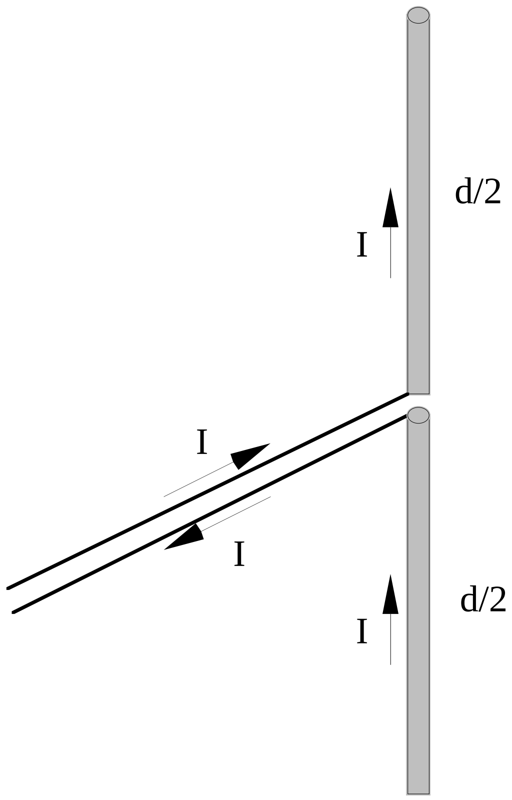
$$\boxed{\mathbf{B}(\mathbf{r},t) = \left(\frac{[\dot{p}]}{cr^2} + \frac{[\ddot{p}]}{c^2 r}\right)\sin\theta\mathbf{e}_\varphi}$$

The terms varying as $1/r$ are the radiation fields. The effects of the time dependence of the charge and current densities are two fold; one is the time delay of the response; the other is that time derivatives enter the expressions.

LINEAR ANTENNAS

Real antennas that are used in practice will not fulfill the requirement from the preceding section that the size of the antenna is much smaller than the wavelength. It is of the same size. Thus we have retardation effects within the source region.

Study the center-driven linear antenna in the figure (linear antennas can also be fed from one end). The current is fed by a remote generator via a transmission line, which does not radiate.



Center-driven linear antenna

We assume sinusoidal standing waves with nodes at the outer endpoints

$$\mathbf{J}(\mathbf{r}',t_r)dv' \to I(z',t_r)d\mathbf{z}' = \mathbf{e}_z I_0 e^{-i\omega t_r}\sin k\left(\tfrac{d}{2}-|z'|\right)dz'$$

and the input signal at the center gap is

$$I_{gap}(t_r) = I_0 \sin k\left(\tfrac{d}{2}\right)e^{-i\omega t_r}$$

Note that in the transmission line the currents are equal and opposite while in the two arms of the antenna they are equal and in the same sense.

We will study the radiation fields and therefore only keep terms that varies as $1/R$. We start from the magnetic field, since there is only one term of this kind.

Generalized Coulomb-Faraday law

$$\boxed{\mathbf{E}(\mathbf{r},t) = \int_V \left(\frac{[\rho]\mathbf{e}_R}{R^2} + \frac{[\partial\rho/\partial t]\mathbf{e}_R}{cR} - \frac{[\partial \mathbf{J}/\partial t]}{c^2 R} \right) dv'}$$

Generalized Biot-Savart law

$$\boxed{\mathbf{B}(\mathbf{r},t) = \int_V \left(\frac{[\mathbf{J}]\times \mathbf{e}_R}{cR^2} + \frac{[\partial \mathbf{J}/\partial t]\times \mathbf{e}_R}{c^2 R} \right) dv'}$$

We get

$$\mathbf{B}_{rad}(\mathbf{r},t) = \int_V \frac{[\partial \mathbf{J}/\partial t]\times \mathbf{e}_R}{c^2 R} dv' \rightarrow$$

$$\boxed{\mathbf{B}_{rad}(\mathbf{r},t) = -(\mathbf{e}_z \times \mathbf{n}) \frac{i\omega}{c^2 r} \int I(z',t_r)dz'}$$

where we have used

$$R \equiv |\mathbf{r}-\mathbf{r}'| \rightarrow r \quad ; \quad \mathbf{e}_R \rightarrow \mathbf{n} \equiv \mathbf{r}/r$$

i.e., used the so called *paraxial* approximation

$$r >> \begin{cases} \lambda & (\textit{radiation} \text{ limit}) \\ d & (\textit{paraxial} \text{ limit}) \end{cases}$$

We see that the integral above can be viewed as the summation of contributions from all elements of the antenna. These contributions have all different phases due to the dependence on the retarded times. Let us make this explicit

$$
\begin{aligned}
B_{rad} = \left(\mathbf{B}_{rad}\right)_{\varphi} &= -\sin\theta \frac{i\omega}{c^2 r}\int I(z')e^{-i\omega t_r}dz' \\
&= -\sin\theta \frac{i\omega}{c^2 r}\int I(z')e^{-i\omega\left(t-|\mathbf{r}-\mathbf{r}'|/c\right)}dz' \\
&= -\sin\theta \frac{i\omega}{c^2 r}e^{-i\omega t}\int I(z')e^{i\omega|\mathbf{r}-\mathbf{r}'|/c}dz' \\
&= -\sin\theta \frac{i\omega}{c^2 r}e^{-i\omega t}\int I(z')e^{ik|\mathbf{r}-\mathbf{r}'|}dz'
\end{aligned}
$$

The phase is expanded in the small parameter r'/r

$$
\begin{aligned}
|\mathbf{r}-\mathbf{r}'| &= \sqrt{r^2 - 2\mathbf{r}\cdot\mathbf{r}' + r'^2} \\
&= r\left[1 - \tfrac{r'}{r}\cos\theta + \tfrac{r'^2}{2r^2}\sin^2\theta + \cdots\right]
\end{aligned}
$$

The quadratic term and higher order terms may be neglected if

$$
r >> \frac{d^2}{8\lambda} \quad (\textit{Fraunhofer}\text{ limit})
$$

In this limit we have

$$
d << \sqrt{\lambda r} << r
$$

So in the Fraunhofer limit we have

$$
\begin{aligned}
B_{rad} &= -\sin\theta \frac{i\omega I_0}{c^2 r}e^{i(kr-\omega t)}\int_{-d/2}^{d/2}\sin k\left(\tfrac{d}{2} - |z'|\right)e^{ikz'\cos\theta}dz' \\
&= -2\sin\theta \frac{i\omega I_0}{c^2 r}e^{i(kr-\omega t)}\int_{0}^{d/2}\sin k\left(\tfrac{d}{2} - z'\right)\cos(kz'\cos\theta)dz'
\end{aligned}
$$

$$= -\sin\theta \frac{i\omega I_0}{c^2 r} e^{i(kr-\omega t)} \int_0^{d/2} \left\{ \sin k\left[\tfrac{d}{2} - z'(1-\cos\theta)\right] + \sin k\left[\tfrac{d}{2} - z'(1+\cos\theta)\right] \right\} dz'$$

$$= -\sin\theta \frac{i\omega I_0}{c^2 r} e^{i(kr-\omega t)} \left[\cos\left(\tfrac{kd}{2}\cos\theta\right) - \cos\left(\tfrac{kd}{2}\right)\right] \left[\frac{1}{k(1-\cos\theta)} + \frac{1}{k(1+\cos\theta)}\right]$$

$$= -\frac{2I_0}{cr} i e^{i(kr-\omega t)} \left(\frac{\cos\left(\tfrac{kd}{2}\cos\theta\right) - \cos\left(\tfrac{kd}{2}\right)}{\sin\theta} \right)$$

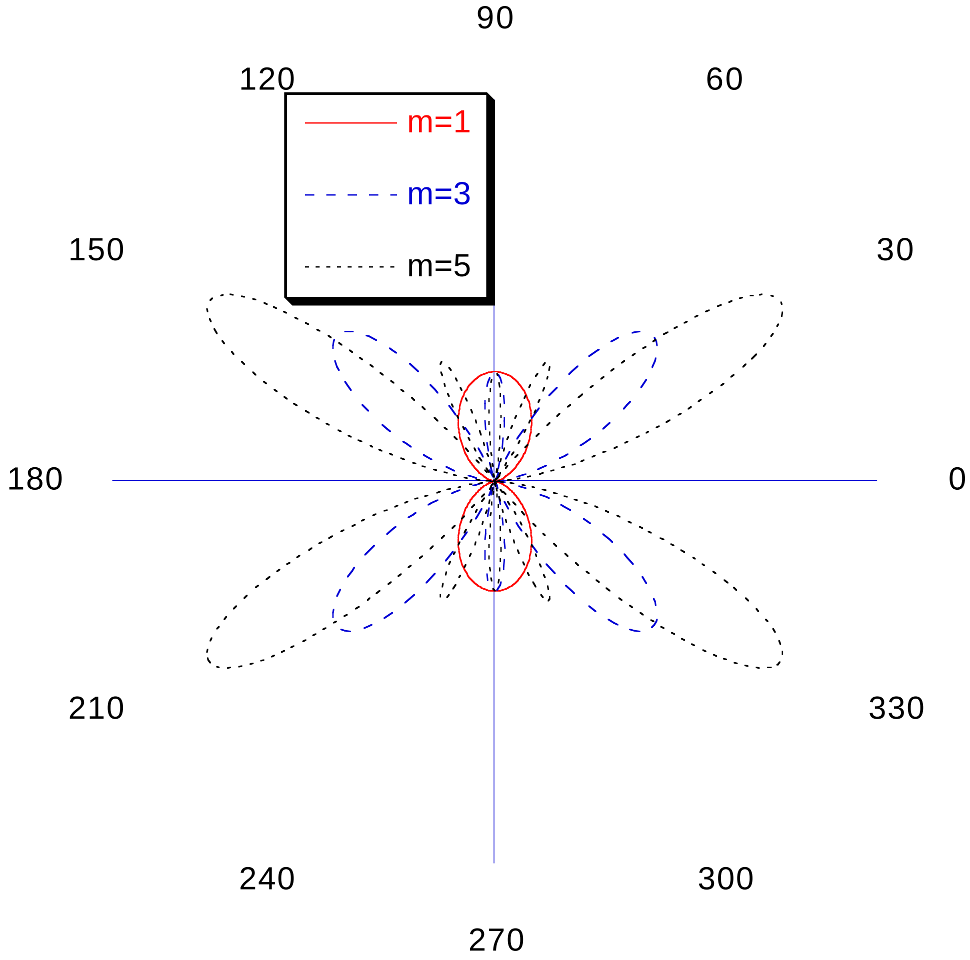
and the time-average power radiated into unit solid angle is

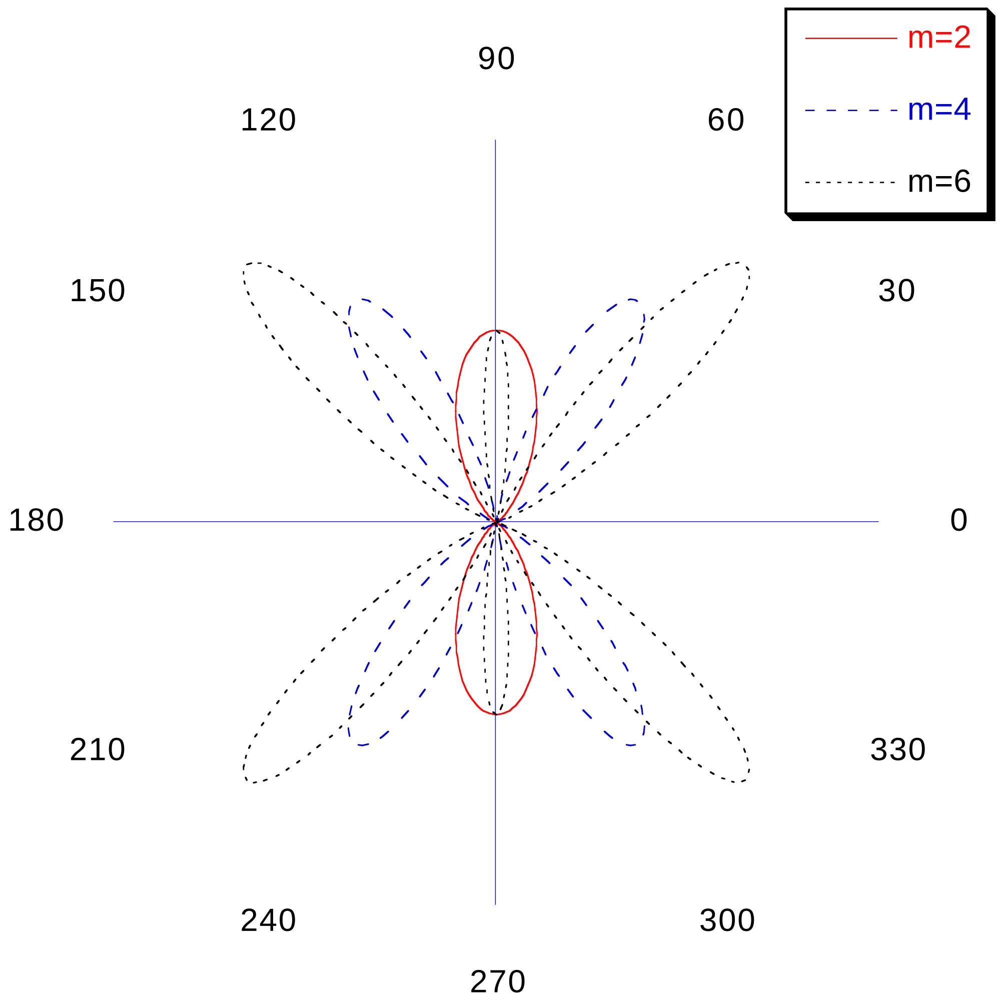
$$\left\langle \frac{dP}{d\Omega} \right\rangle = r^2 \langle \mathbf{S}_{rad} \rangle \cdot \mathbf{n} = \frac{cr^2}{4\pi} \langle B_{rad}{}^2 \rangle$$

or

$$\boxed{\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{I_0{}^2}{2\pi c} \left(\frac{\cos\left(\tfrac{kd}{2}\cos\theta\right) - \cos\left(\tfrac{kd}{2}\right)}{\sin\theta} \right)^2}$$

Of special interest are antenna where d is an integral number m of half wavelengths. Then $kd = m\pi$.





90
120
60
150
30
180
0
210
330
240
300
270
m=2
m=4
m=6

ANTENNA DIRECTIVITY AND EFFECTIVE AREA

Transmitting antenna:

The *directivity* or *gain* of an antenna is defined as the ratio of the *maximum* value of the power radiated per-unit-solid-angle to the *average* power radiated per-unit-solid-angle.

$$\boxed{G = \frac{\langle dP/d\Omega \rangle_{\max}}{\langle P \rangle / 4\pi}}$$

Antenna Gain or *Directivity*

Receiving antenna:

The *effective area* of a receiving antenna is

$$\boxed{A = G\pi\bar{\lambda}^2}$$

Effective area of receiving antenna.

Antenna-to-antenna communication link:

$$\boxed{\frac{P_{out}}{P_{in}} = G_t G_r \left(\frac{\lambda}{4\pi r} \right)^2 = \frac{A_t A_r}{\lambda^2 r^2}}$$

Friis transmission formula

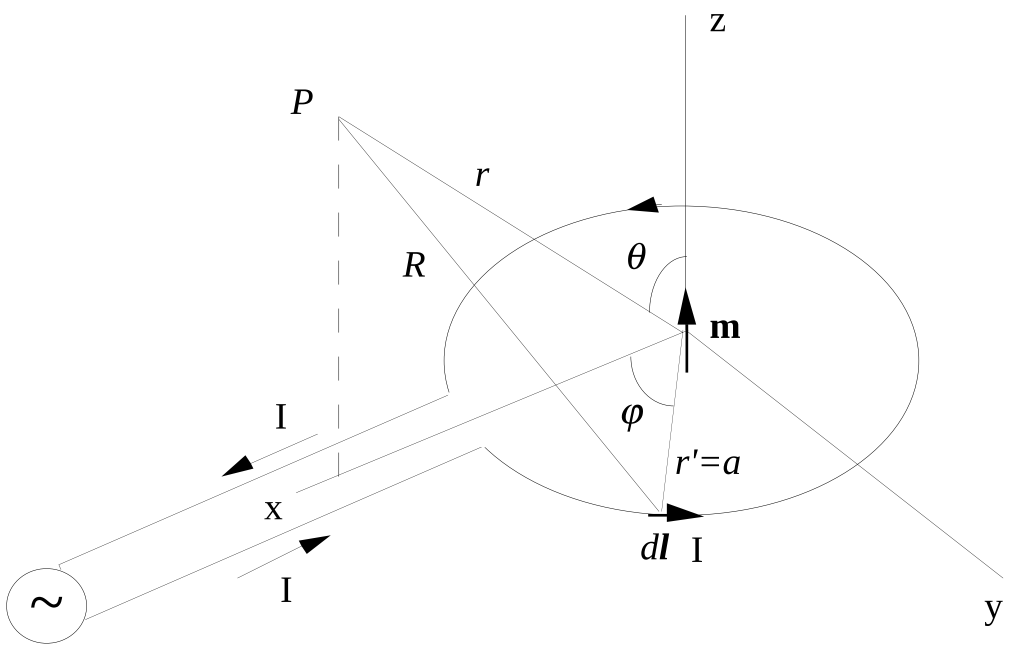
MAGNETIC DIPOLE RADIATION

A general distribution of charge and currents possesses both electric and magnetic multipole moments. The higher order electric multipole radiation drops with order.

Here we will only study the magnetic dipole radiation and see how it compares with its electric counterpart.

We will study the circular loop in the figure, below, and use the same hierarchy as for the radiation field of the electric dipole:

$d << \lambda << r.$



We start from the radiation fields of the equations:

$$\mathbf{E} = \int_V \left(\frac{[\rho]\mathbf{e}_R}{R^2} + \frac{[\partial\rho/\partial t]\mathbf{e}_R}{cR} - \frac{[\partial\mathbf{J}/\partial t]}{c^2 R} \right) dv'$$

$$\mathbf{B}(\mathbf{r},t) = \int_V \left(\frac{[\mathbf{J}]\times\mathbf{e}_R}{cR^2} + \frac{[\partial\mathbf{J}/\partial t]\times\mathbf{e}_R}{c^2 R} \right) dv'$$

which are

$$\mathbf{E}_{rad} = -\int_V \frac{[\partial\mathbf{J}/\partial t]}{c^2 R} dv'$$

$$\mathbf{B}_{rad} = \int_V \frac{[\partial\mathbf{J}/\partial t]\times\mathbf{e}_R}{c^2 R} dv'$$

In our example we have

$$I(t) = I_0 e^{-i\omega t'} \rightarrow I_0 \cos(\omega t')$$

and

$$\mathbf{J}dv' \rightarrow I d\boldsymbol{l}$$

Thus we have

$$\mathbf{E}_{rad} \rightarrow \frac{\omega I_0}{c^2 r} \oint_\Gamma \sin\omega(t - R/c)\, d\boldsymbol{l}$$

We have made the approximation

$$R \equiv |\mathbf{r} - \mathbf{r}'| \rightarrow r$$

in the denominator but cannot do the same in the argument of the sine function. If we do we get a null result. This means that we cannot neglect retardation effects within the source even if $a << \lambda$.

We choose our origin at the center of the loop and let the observation point be above the *x*-axis (just for convenience).

$$\mathbf{r} = r(\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}})$$
$$\mathbf{r}' = a(\cos\varphi\hat{\mathbf{x}} + \sin\varphi\hat{\mathbf{y}})$$

and

$$R = \left(r^2 - 2\mathbf{r}\cdot\mathbf{r}' + r'^2\right)^{1/2} \to r\left(1 - \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2} + \cdots\right)$$
$$\to r\left(1 - \frac{a}{r}\sin\theta\cos\varphi + \cdots\right)$$

So the sine function is to lowest order is

$$\begin{aligned}\sin[\omega(t - R/c)] \to \sin&\left[\omega(t - r/c) + \frac{\omega a}{c}\sin\theta\cos\varphi\right]\\ &= \sin[\omega(t - r/c)]\cos\left[\frac{\omega a}{c}\sin\theta\cos\varphi\right]\\ &+ \cos[\omega(t - r/c)]\sin\left[\frac{\omega a}{c}\sin\theta\cos\varphi\right]\\ &\to \sin[\omega(t - r/c)]\\ &+ \cos[\omega(t - r/c)]\left(\frac{\omega a}{c}\sin\theta\cos\varphi\right)\end{aligned}$$

We furthermore need

$$d\boldsymbol{l} = a\,d\varphi(-\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}})$$

This gives

$$
\begin{aligned}
\mathbf{E}_{rad} &= \frac{\omega I_0}{c^2 r}\int_0^{2\pi}\left\{\sin[\omega(t-r/c)]+\cos[\omega(t-r/c)]\right. \\
&\qquad \left.\times \omega a \sin\theta\cos\varphi/c\right\} a\,d\varphi(-\sin\varphi\hat{\mathbf{x}}+\cos\varphi\hat{\mathbf{y}}) \\
&= \frac{a\omega I_0}{c^2 r}\hat{\mathbf{y}}\int_0^{2\pi}\left\{\cos[\omega(t-r/c)]\omega a\sin\theta\cos^2\varphi/c\right\}d\varphi \\
&= \frac{\pi a^2\omega^2 I_0}{c^3 r}\sin\theta\cos[\omega(t-r/c)]\hat{\mathbf{y}} \\
&\rightarrow \frac{\pi a^2\omega^2 I_0}{c^3 r}\sin\theta e^{i(kr-\omega t)}\mathbf{e}_\varphi
\end{aligned}
$$

(Note that I get a different sign compared to the text book)

Note that $\hat{\mathbf{y}} \rightarrow \mathbf{e}_\varphi$ generalizes the result to a general field point. We forced the field point to be above the *x*-axis.

The magnetic radiation field is then

$$
\begin{aligned}
\mathbf{B}_{rad} &= -\mathbf{E}_{rad}\times\mathbf{e}_r \\
&= -\frac{\pi a^2\omega^2 I_0}{c^3 r}\sin\theta e^{i(kr-\omega t)}\mathbf{e}_\varphi\times\mathbf{e}_r \\
&= -\frac{\pi a^2\omega^2 I_0}{c^3 r}\sin\theta e^{i(kr-\omega t)}\mathbf{e}_\theta = \left[m=\frac{\pi a^2 I}{c}\right] = \\
&= \frac{\ddot{m}e^{i(kr)}}{c^2 r}\sin\theta\mathbf{e}_\theta = \frac{\ddot{m}e^{i(\omega r/c)}}{c^2 r}\sin\theta\mathbf{e}_\theta = \frac{[\ddot{m}]}{c^2 r}\sin\theta\mathbf{e}_\theta
\end{aligned}
$$

and

$$\boxed{\mathbf{E}_{rad} = -\frac{[\ddot{m}]}{c^2 r}\sin\theta\mathbf{e}_{\varphi}}$$ *Magnetic dipole radiation*

$$\boxed{\mathbf{B}_{rad} = \frac{[\ddot{m}]}{c^2 r}\sin\theta\mathbf{e}_{\theta}}$$ *Magnetic dipole radiation*

to be compared to the results from the electric dipole radiation:

$$\boxed{\mathbf{E}_{rad} = \frac{[\ddot{p}]}{c^2 r}\sin\theta\mathbf{e}_{\theta}}$$ *Electric dipole radiation*

$$\boxed{\mathbf{B}_{rad} = \frac{[\ddot{p}]}{c^2 r}\sin\theta\mathbf{e}_{\varphi}}$$ *Electric dipole radiation*

This shows that the radiation from electric and magnetic dipoles are *duals*.

We have for the angular distribution of radiated power:

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{c}{8\pi}\left(\mathbf{E}_{rad}\times\mathbf{B}_{rad}^{*}\right)r^2\mathbf{e}_r = \frac{\left\langle\left[\ddot{m}^2\right]\right\rangle}{4\pi c^3}\sin^2\theta$$
$$= \frac{\omega^4 I_0^2 S^2}{8\pi c^5}\sin^2\theta$$

and total power:

$$\langle P\rangle = \frac{\omega^4 I_0^2 S^2}{3c^5} = \frac{\pi^6 I_0^2}{3c}\left(\frac{d}{\lambda}\right)^4$$

The magnetic dipole radiator is less effective than the electric dipole counterpart. This can be expressed in terms of the radiation resistance:

$$\langle P\rangle = \frac{\pi^6 I_0^2}{3c}\left(\frac{d}{\lambda}\right)^4 = \frac{1}{2}R_{rad}I_0^2$$

In the figure below, we have compared the radiation resistance for the magnetic dipole radiator, the electric dipole radiator and the center-fed linear antenna.

The first two are only valid for very small d/λ values, where d is the length of the electric dipole and the diameter of the magnetic dipole, respectively.

For a center-fed linear antenna with even m-values the driving-point resistance diverges and one has to use tricks to make the antenna function, like widening the gap.