RETARDED POTENTIALS AND FIELDS AND RADIATION BY CHARGED PARTICLES

Retarded potentials

In the case of static potentials we found that the potentials could be obtained from the charge and current densities according to these relations:

$$\Phi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dv'$$

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dv'$$

When we have time dependent densities and resulting fields we have to take the finite speed of the fields into account. There are *retardation effects*. The fields in a point **r** at a time *t* depends on the densities in a point **r**' at an earlier time *t*',

$$t' = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}$$

Now, for the potentials, the retardation depends on the choice of gauge. In Lorentz gauge both potentials are retarded. In Coulomb gauge only the vector potential is retarded, the scalar potential is instantaneous.

We have:

$$\boxed{\begin{aligned} \Phi(\mathbf{r},t) &= \int_V \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} dv' \\ \mathbf{A}(\mathbf{r},t) &= \frac{1}{c}\int_V \frac{\mathbf{J}_\perp\left(\mathbf{r}', t - |\mathbf{r}-\mathbf{r}'|/c\right)}{|\mathbf{r}-\mathbf{r}'|} dv' \end{aligned}}$$

Coulomb Gauge

$$\boxed{\begin{aligned}\Phi(\mathbf{r},t) &= \int_V \frac{\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}dv' \\ \mathbf{A}(\mathbf{r},t) &= \frac{1}{c}\int_V \frac{\mathbf{J}(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}dv'\end{aligned}}$$

Lorentz Gauge

Static Electromagnetism

Let us first rederive the potentials in the static field case. Then we have

$$\nabla^2\Phi(\mathbf{r}) = -4\pi\rho(\mathbf{r})$$
$$\nabla^2\mathbf{A}(\mathbf{r}) = -\frac{4\pi}{c}\mathbf{J}(\mathbf{r})$$

Both satisfy the same type of differential equation. We can concentrate on one of them, the first say.

$$(i\mathbf{q})^2\Phi(\mathbf{q},\omega) = -4\pi\rho(\mathbf{q},\omega)$$

$$-q^2\Phi(\mathbf{q},\omega) = -4\pi\rho(\mathbf{q},\omega)$$

Thus,

$$\Phi(\mathbf{q},\omega) = \frac{4\pi}{q^2}\rho(\mathbf{q},\omega)$$

or

$$\Phi(\mathbf{q},\omega) = v(\mathbf{q},\omega)\,\rho(\mathbf{q},\omega)/e^2 \quad ; \quad v(\mathbf{q},\omega) = \frac{4\pi e^2}{q^2}$$

Thus, we obtain the following convolution integral relating the potential to the charge distribution:

$$\Phi(\mathbf{r},t) = \frac{1}{e^2}\iint d^3r'\,dt'\,v(\mathbf{r}-\mathbf{r}',t-t')\rho(\mathbf{r}',t')$$

The function $v(\mathbf{r},t)$ is the Coulomb potential. It represents the potential energy between two electrons a distance r apart. It is obtained from its Fourier transform above using the inverse Fourier transform. Note that its Fourier transform is independent of ω. This means that the function will be a delta function of time. The Coulomb potential is instantaneous.

$$\begin{aligned}
v(\mathbf{r},t) &= \int \frac{d^3q}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} v(\mathbf{q},\omega) e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)} \\
&= \int \frac{d^3q}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{4\pi e^2}{q^2} e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)} \\
&= \int \frac{d^3q}{(2\pi)^3} \frac{4\pi e^2}{q^2} e^{i(\mathbf{q}\cdot\mathbf{r})} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i(-\omega t)} \\
&= \delta(t) \int_0^{\infty} \frac{dq}{(2\pi)^3} 2\pi q^2 \int_{-1}^{1} dx \frac{4\pi e^2}{q^2} e^{iqrx} \\
&= \delta(t) \int_0^{\infty} \frac{dq}{(2\pi)^3} 2\pi q^2 \frac{4\pi e^2}{q^2} \frac{1}{iqr} e^{iqrx} \Big|_{-1}^{1} \\
&= \delta(t) \frac{2e^2}{\pi r} \int_0^{\infty} dq \frac{\sin(qr)}{q} \\
&= \delta(t) \frac{2e^2}{\pi r} \frac{\pi}{2} = \delta(t) \frac{e^2}{r}
\end{aligned}$$

Thus

$$\begin{aligned}
\Phi(\mathbf{r},t) &= \frac{1}{e^2} \iint d^3r'\, dt' \delta(t-t') \frac{e^2}{|\mathbf{r}-\mathbf{r}'|} \rho(\mathbf{r}',t') \\
&= \int d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \rho(\mathbf{r}',t)
\end{aligned}$$

or

$$\Phi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dv'$$

since the charge distribution was assumed to be static. This agrees with equations (4.59) and (8.1).

If we repeat the same procedure for the vector potential we get

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dv'$$

Dynamic Electromagnetism

LORENTZ GAUGE

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -4\pi\rho$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J}$$

Both potentials satisfy the same type of equation, the wave equation. We concentrate on the equation for the scalar potential

$$(i\mathbf{q})^2\Phi(\mathbf{q},\omega) - \frac{(-i\omega)^2}{c^2}\Phi(\mathbf{q},\omega) = -4\pi\rho(\mathbf{q},\omega)$$

Rearrangement gives

$$\Phi(\mathbf{q},\omega) = u(\mathbf{q},\omega)\,\rho(\mathbf{q},\omega)\big/e^2 \quad ; \quad u(\mathbf{q},\omega) = \frac{4\pi e^2}{q^2}\frac{1}{1-(\omega/cq)^2}$$

Before we continue let us return to the wave equation above. It is quadratic in the time derivative, which means that it describes a system that has time inversion symmetry.

This means that if we are not careful we will find that the potential at a certain time will depend on the charge distribution not only at preceding times but also at future times.

This is in conflict with causality. We believe that the response always comes after the perturbation. This is achieved by adding a positive infinitesimal imaginary part to the frequency in the response function, $u(\mathbf{q},\omega)$ in this case:

$$\begin{aligned}
u(\mathbf{q},\omega) &= \frac{2\pi e^2}{q^2}\left[\frac{1}{\omega/cq+i\delta+1}-\frac{1}{\omega/cq+i\delta-1}\right]\\
&= \frac{2\pi e^2}{q^2}\left[\frac{1}{\omega/cq+1}-\frac{1}{\omega/cq-1}\right]\\
&+\frac{2\pi^2 e^2 i}{q^2}\left[-\delta(\omega/cq+1)+\delta(\omega/cq-1)\right]
\end{aligned}$$

The poles of retarded correlations functions are always below the real frequency axis in the complex frequency plane. If we do not add these imaginary parts half of the response to a perturbation will be in the form of a retarded response and half in the form of an advanced response.

Now, we have

$$u(\mathbf{r},t)=\int\frac{d^3q}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}u(\mathbf{q},\omega)e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}$$

or

$$\begin{aligned}
u(\mathbf{r},t) &= u_1(\mathbf{r},t)+u_2(\mathbf{r},t)\\
&= \int\frac{d^3q}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left[u_1(\mathbf{q},\omega)+u_2(\mathbf{q},\omega)\right]e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}
\end{aligned}$$

where

$$u_1(\mathbf{q},\omega)=\frac{2\pi e^2}{q^2}\left[\frac{1}{\omega/cq+1}-\frac{1}{\omega/cq-1}\right]\ ;$$

$$u_2(\mathbf{q},\omega)=\frac{2\pi^2 e^2 i}{q^2}\left[-\delta(\omega/cq+1)+\delta(\omega/cq-1)\right]$$

Thus

$$
\begin{aligned}
u_1(\mathbf{r},t) &= \int \frac{d^3q}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{2\pi e^2}{q^2} \left[\frac{1}{\omega/cq+1} - \frac{1}{\omega/cq-1} \right] e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)} \\
&= [\omega \to \omega cq] \\
&= \int \frac{d^3q}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{2\pi e^2 cq}{q^2} \left[\frac{1}{\omega+1} - \frac{1}{\omega-1} \right] e^{i(\mathbf{q}\cdot\mathbf{r}-cq\omega t)} \\
&= \frac{ce^2}{4\pi^2} \int_0^{\infty} dq q^2 \frac{1}{q} \int_{-1}^{1} dx e^{iqrx} \int_{-\infty}^{\infty} d\omega \left[\frac{1}{\omega+1} - \frac{1}{\omega-1} \right] e^{-icq\omega t} \\
&= \frac{ce^2}{4\pi^2} \int_0^{\infty} dq q 2 \frac{\sin(qr)}{qr} \int_{-\infty}^{\infty} d\omega \frac{1}{\omega} e^{-icq\omega t} \left[e^{icqt} - e^{-icqt} \right] \\
&= \frac{ce^2}{2\pi^2 r} \int_0^{\infty} dq \sin(qr) 2i \sin(cqt) \int_0^{\infty} d\omega \frac{1}{\omega} \left[-2i \sin(cq\omega t) \right] \\
&= \frac{2ce^2}{\pi^2 r} \int_0^{\infty} dq \left[\tfrac{1}{2} \cos(qr-cqt) - \tfrac{1}{2} \cos(qr+cqt) \right] \frac{\pi}{2} sign(cqt) \\
&= \frac{ce^2}{4\pi r} sign(t) \int_{-\infty}^{\infty} dq \left[\cos(qr-cqt) - \cos(qr+cqt) \right] \\
&= \frac{ce^2}{4\pi r} sign(t) \left[2\pi\delta(r-ct) - 2\pi\delta(r+ct) \right] \\
&= \frac{ce^2}{2r} \left[\delta(r-ct) + \delta(r+ct) \right] \\
&= \frac{e^2}{2r} \left[\delta(t-r/c) + \delta(t+r/c) \right]
\end{aligned}
$$

and

$$
\begin{aligned}
u_2(\mathbf{r},t) &= \int \frac{d^3q}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{2\pi^2 e^2 i}{q^2} \left[-\delta(\omega/cq+1)+\delta(\omega/cq-1)\right] e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)} \\
&= [\omega \to \omega c q] \\
&= \int \frac{d^3q}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{2\pi^2 e^2 icq}{q^2} \left[-\delta(\omega+1)+\delta(\omega-1)\right] e^{i(\mathbf{q}\cdot\mathbf{r}-cq\omega t)} \\
&= i\frac{ce^2}{4\pi} \int_0^{\infty} dq q^2 \frac{1}{q} \int_{-1}^{1} dx e^{iqrx} \left[-e^{icqt}+e^{-icqt}\right] \\
&= \frac{ice^2}{4\pi} \int_0^{\infty} dq q 2 \frac{\sin(qr)}{qr} \left[-2i\sin(cqt)\right] \\
&= \frac{ce^2}{\pi r} \int_0^{\infty} dq \sin(qr)\sin(cqt) \\
&= \frac{ce^2}{4\pi r} \int_{-\infty}^{\infty} dq \left[\cos(qr-cqt)-\cos(qr+cqt)\right] \\
&= \frac{ce^2}{4\pi r} \left[2\pi\delta(r-ct)-2\pi\delta(r+ct)\right] = \\
&= \frac{ce^2}{2r} \left[\delta(r-ct)-\delta(r+ct)\right] \\
&= \frac{e^2}{2r} \left[\delta(t-r/c)-\delta(t+r/c)\right]
\end{aligned}
$$

Thus

$$
\begin{aligned}
u(\mathbf{r},t) &= u_1(\mathbf{r},t) + u_2(\mathbf{r},t)\\
&= \frac{e^2}{2r}\left[\delta(t-r/c)+\delta(t+r/c)\right] + \frac{e^2}{2r}\left[\delta(t-r/c)-\delta(t+r/c)\right]\\
&= \frac{e^2}{r}\delta(t-r/c)
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi(\mathbf{r},t) &= \frac{1}{e^2}\iint d^3r'\, dt' u(\mathbf{r}-\mathbf{r}',t-t')\rho(\mathbf{r}',t')\\
&= \frac{1}{e^2}\iint d^3r'\, dt' \frac{e^2}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t-t'-|\mathbf{r}-\mathbf{r}'|/c\right)\rho(\mathbf{r}',t')\\
&= \iint d^3r'\, dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t-t'-|\mathbf{r}-\mathbf{r}'|/c\right)\rho(\mathbf{r}',t')\\
&= \int d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\rho\left(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c\right)
\end{aligned}
$$

This is equation (8.4).

$$
\Phi(\mathbf{r},t) = \int_V \frac{\rho\left(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c\right)}{|\mathbf{r}-\mathbf{r}'|}dv
$$

Treating the vector potential in the same fashion we arrive at

$$
\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int_V \frac{\mathbf{J}\left(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c\right)}{|\mathbf{r}-\mathbf{r}'|}dv'
$$

Discussion

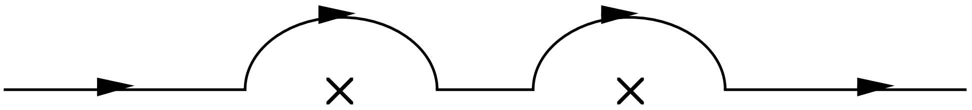
We found that adding a positive infinitesimal imaginary part to the angular frequency of the response function rendered it causal, i.e., it only transferred information from the past. The result was the retarded version of the function.

Had we added negative imaginary parts instead, only future events would contribute to the potentials and fields and the response function would be the advanced version of the function.

An equivalent, alternative way to handle things would be to prescribe that we made small detours around the poles on the real axes, detours in the upper half of the complex frequency plane. Our correlation function had two singularities, or poles, on the real frequency axis, at $\omega = \pm cq$.

To get the causal result one moves the poles down a little bit or one makes the detours.

or



In our derivation we have assumed that the charge and current densities are in vacuum. If we had embedded them in a medium with a dielectric function, we had not had to add these infinitesimal imaginary parts to the frequency. The dielectric function had provided these. The poles had ended up below the real axis. It is just when we treat the idealized system with no dissipation that we have to artificially add these infinitesimal imaginary parts.

COULOMB GAUGE

$$\nabla^2 \Phi = -4\pi\rho$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J}_\perp$$

The differential equation for the scalar potential is the same Poisson equation as in the static electromagnetism and the equation for the vector potential is the same as in Lorentz gauge except for the the right hand side now contains the transverse component, only, of the current density.

Thus we immediately find:

$$\Phi(\mathbf{r},t) = \int_V \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}dv'$$

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int_V \frac{\mathbf{J}_\perp(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}dv'$$

RETARDED FIELDS

All the fields will be retarded. These are independent of choice of gauge. We will in what follows indicate that the time argument of an expression is the retarded one by enclosing the expression in a square bracket.

Let us first start from Lorentz gauge:

$$\Phi(\mathbf{r},t) = \int_V \frac{[\rho(\mathbf{r}')]}{|\mathbf{r}-\mathbf{r}'|} dv'$$

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int_V \frac{[\mathbf{J}(\mathbf{r}')]}{|\mathbf{r}-\mathbf{r}'|} dv'$$

The electric field is obtained from

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

We find

$$\mathbf{E} = \int_V \left(-\nabla \frac{[\rho(\mathbf{r}')]}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{c^2}\frac{\partial}{\partial t}\frac{[\mathbf{J}(\mathbf{r}')]}{|\mathbf{r}-\mathbf{r}'|} \right) dv'$$

$$= \int_V \left(\frac{[\rho]\mathbf{e}_R}{R^2} + \frac{[\partial\rho/\partial t]\mathbf{e}_R}{cR} - \frac{[\partial\mathbf{J}/\partial t]}{c^2 R} \right) dv'$$

where

$$\mathbf{R} = \mathbf{r} - \mathbf{r}' \quad ; \quad R = |\mathbf{r}-\mathbf{r}'|$$

The magnetic induction is obtained from

$$\mathbf{B} = \nabla \times \mathbf{A}$$

We find with the help from the relation

$$\nabla \times (\psi \mathbf{A}) = \psi \nabla \times \mathbf{A} - \mathbf{A} \times (\nabla \psi)$$

$$
\begin{aligned}
\mathbf{B}(\mathbf{r},t) &= \int_V \nabla\times\frac{[\mathbf{J}(\mathbf{r}')]}{cR}dv' \\
&= \int_V \left(\frac{1}{cR}\nabla\times[\mathbf{J}(\mathbf{r}')]-[\mathbf{J}(\mathbf{r}')]\times\left(\nabla\left(\frac{1}{cR}\right)\right)\right)dv' \\
&= \int_V \left(\frac{[\partial\mathbf{J}/\partial t]\times\mathbf{e}_R}{c^2R}+\frac{[\mathbf{J}]\times\mathbf{e}_R}{cR^2}\right)dv'
\end{aligned}
$$

Generalized Coulomb-Faraday law

$$
\boxed{\mathbf{E}(\mathbf{r},t) = \int_V \left(\frac{[\rho]\mathbf{e}_R}{R^2}+\frac{[\partial\rho/\partial t]\mathbf{e}_R}{cR}-\frac{[\partial\mathbf{J}/\partial t]}{c^2R}\right)dv'}
$$

Generalized Biot-Savart law

$$
\boxed{\mathbf{B}(\mathbf{r},t) = \int_V \left(\frac{[\mathbf{J}]\times\mathbf{e}_R}{cR^2}+\frac{[\partial\mathbf{J}/\partial t]\times\mathbf{e}_R}{c^2R}\right)dv'}
$$

We see that all terms are retarded.

These results are useful when we describe macroscopic charge and current distributions. When we are dealing with moving point charges there are better ways. This we will deal with in next section.

Before we continue let us make sure that we obtain the same fields when using the potentials in Coulomb gauge.

In Coulomb gauge we have the instantaneous scalar potential and the transverse, retarded, vector potential

$$\Phi(\mathbf{r},t) = \int_V \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}dv'$$

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int_V \frac{[\mathbf{J}_\perp(\mathbf{r}')]}{|\mathbf{r}-\mathbf{r}'|}dv$$

It seems a little bit odd that these should result in retarded fields, but they do. To see this we use Fourier transforms.

$$\boxed{\begin{aligned}\Phi(\mathbf{q},\omega) &= \frac{4\pi\rho(\mathbf{q},\omega)}{q^2}\frac{1}{1-(\omega/cq)^2}\\ \mathbf{A}(\mathbf{q},\omega) &= \frac{4\pi\mathbf{J}(\mathbf{q},\omega)}{cq^2}\frac{1}{1-(\omega/cq)^2}\end{aligned}}$$

Lorentz gauge

The last, frequency dependent, factor makes, as we have seen already, the fields retarded.

In Coulomb gauge we have

$$\boxed{\begin{aligned}\Phi(\mathbf{q},\omega) &= \frac{4\pi\rho(\mathbf{q},\omega)}{q^2}\\ \mathbf{A}(\mathbf{q},\omega) &= \frac{4\pi\mathbf{J}_\perp(\mathbf{q},\omega)}{cq^2}\frac{1}{1-(\omega/cq)^2}\end{aligned}}$$

Coulomb gauge

The electric field is always given by

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}$$

This leads to the following relation between the Fourier transforms

$$\mathbf{E}(\mathbf{q},\omega) = -i\mathbf{q}\Phi(\mathbf{q},\omega) + \frac{i\omega}{c}\mathbf{A}(\mathbf{q},\omega)$$

In Lorentz gauge we have

$$\boxed{\mathbf{E}(\mathbf{q},\omega) = -i\mathbf{q}\frac{4\pi\rho(\mathbf{q},\omega)}{\left[q^2-(\omega/c)^2\right]} + \frac{i\omega}{c^2}\frac{4\pi\mathbf{J}(\mathbf{q},\omega)}{\left[q^2-(\omega/c)^2\right]}} \qquad \textit{Lorentz gauge}$$

and in Coulomb gauge:

$$\boxed{\mathbf{E}(\mathbf{q},\omega) = -i\mathbf{q}\frac{4\pi\rho(\mathbf{q},\omega)}{q^2} + \frac{i\omega}{c^2}\frac{4\pi\mathbf{J}_\perp(\mathbf{q},\omega)}{\left[q^2-(\omega/c)^2\right]}} \qquad \textit{Coulomb gauge}$$

To see that this is the same we add the longitudinal current density in the last term and subtract the resulting term

$$\mathbf{E}(\mathbf{q},\omega) = -i\mathbf{q}\frac{4\pi\rho(\mathbf{q},\omega)}{q^2} - \frac{i\omega}{c^2}\frac{4\pi\mathbf{J}_{//}(\mathbf{q},\omega)}{\left[q^2-(\omega/c)^2\right]} + \frac{i\omega}{c^2}\frac{4\pi\mathbf{J}(\mathbf{q},\omega)}{\left[q^2-(\omega/c)^2\right]}$$

Next step is to use the equation of continuity

$$\nabla\cdot\mathbf{J} + \frac{\partial\rho}{\partial t} = 0$$

and its Fourier transformed version:

$$i\mathbf{q}\cdot\mathbf{J}(\mathbf{q},\omega) = i\mathbf{q}\cdot\mathbf{J}_{//}(\mathbf{q},\omega) = i\omega\rho(\mathbf{q},\omega)$$

and

$$q^2\mathbf{J}_{//}(\mathbf{q},\omega)=\mathbf{q}\left[\mathbf{q}\cdot\mathbf{J}_{//}(\mathbf{q},\omega)\right]=\mathbf{q}\omega\rho(\mathbf{q},\omega)$$

or

$$\mathbf{J}_{//}(\mathbf{q},\omega)=\mathbf{q}\frac{\omega}{q^2}\rho(\mathbf{q},\omega)$$

We now use this relation to eliminate the longitudinal-current term in our expression.

$$\begin{aligned}\mathbf{E}(\mathbf{q},\omega)&=-i\mathbf{q}\frac{4\pi\rho(\mathbf{q},\omega)}{q^2}-i\mathbf{q}\frac{(\omega/c)^2}{q^2}\frac{4\pi\rho(\mathbf{q},\omega)}{\left[q^2-(\omega/c)^2\right]}\\&\quad+\frac{i\omega}{c^2}\frac{4\pi\mathbf{J}(\mathbf{q},\omega)}{\left[q^2-(\omega/c)^2\right]}\\&=-i\mathbf{q}\frac{4\pi\rho(\mathbf{q},\omega)}{\left[q^2-(\omega/c)^2\right]}+\frac{i\omega}{c^2}\frac{4\pi\mathbf{J}(\mathbf{q},\omega)}{\left[q^2-(\omega/c)^2\right]}\end{aligned}$$

and we are done with the proof for the electric field. The magnetic field is more trivial. We have

$$\mathbf{B}=\nabla\times\mathbf{A}\Rightarrow\mathbf{B}(\mathbf{q},\omega)=i\mathbf{q}\times\mathbf{A}(\mathbf{q},\omega)=i\mathbf{q}\times\mathbf{A}_{\perp}(\mathbf{q},\omega)$$

Thus, only the transverse part of the vector potential is picked out. The transverse part is the same in the Lorentz and Coulomb gauges.

You see that working with the Fourier transforms is a very powerful method.