WAVE GUIDES

We will discuss three types of wave guides:

a) two-conductor transmission lines (radio waves, ~10^8 *Hz*, ~*m*)

b) hollow-conductor wave guides (microwaves, ~10^{10} *Hz*, ~*cm*)

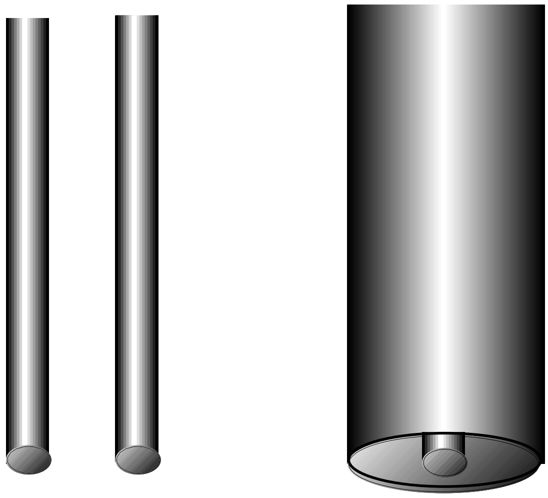
c) optical fibers (near infrared, 10^{14} *Hz*, ~μm)

In common for wave guides is that there is a source of electromagnetic energy, a drain and there in between a geometrical structure that guides the flow of energy from the source to the drain.

Note that the wavelengths are in all these cases much shorter than for ordinary electric circuits where $\nu = 50$ *Hz*, ~10 000 *km*.

The normal or principal mode of the two-conductor transmission line is a *TEM* mode (transverse electric and magnetic). In the other two examples there are no such mode. The mode can be *TE*, *TM* or neither.

TWO-CONDUCTOR TRANSMISSION LINES



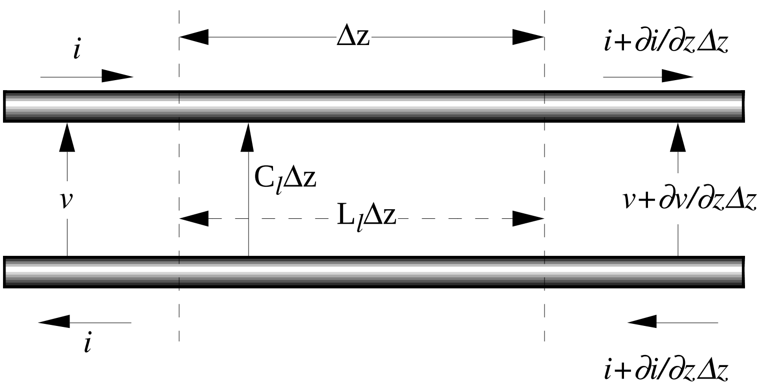
(a) Parallel-wire

(b) Coaxial

These types are recognized from TV-cables.

We assume that the conductors are resistance less, extend indefinitely in the z-direction and are surrounded by air. In this example it is best to describe the system in terms of currents and voltages instead of fields.

We study an element of length Δz.



The geometry of the system is characterized by the *capacitance-per-unit-length* C_l and *inductance-per-unit-length* L_l.

The Capacitance of a pair of conductors is by definition

$$C \equiv \frac{q}{v}$$

and the self-inductance, L, of a circuit loop relates the EMF, ε, induced in the loop to the time-rate-of-change of current by the definition

$$L \equiv -\frac{\varepsilon}{\partial i/\partial t}$$

The first relation gives

$$\Delta v = \frac{-(\partial i/\partial z)\Delta z \Delta t}{C_l \Delta z} \rightarrow \frac{\partial v}{\partial t} = -\frac{1}{C_l}\frac{\partial i}{\partial z}$$

where the numerator is the net amount of charge delivered to the length element of the upper conductor in the time element. The lower conductor receives an equal amount of opposite sign.

Use of the second definition gives

$$\frac{\partial v}{\partial z}\Delta z = -L_l \Delta z \frac{\partial i}{\partial t} \rightarrow \frac{\partial i}{\partial t} = -\frac{1}{L_l}\frac{\partial v}{\partial z}$$

We now take the time derivative of our first relation and the space derivative of our second and then vice versa, eliminate the mixed derivatives, and arrive at

$$\frac{\partial^2 v}{\partial z^2} = L_l C_l \frac{\partial^2 v}{\partial t^2};$$

$$\frac{\partial^2 i}{\partial z^2} = L_l C_l \frac{\partial^2 i}{\partial t^2}$$

The one-dimensional wave equation for the voltage and current.

The potential difference and current travel along the transmission line at the speed

$$c = \frac{1}{\sqrt{L_l C_l}}$$

For loss-free lines this speed is just the speed of light in the surrounding medium.

Impedances and reflection coefficient.

For an infinite lossless transmission line the currents and voltages are in phase and if the voltage is

$$v(z,t) = V_0 e^{i(kz-\omega t)} \quad ; \quad k = \omega/c = \omega\sqrt{L_l C_l}$$

the current is from the differential relations:

$$i(z,t) = \sqrt{\frac{C_l}{L_l}} V_0 e^{i(kz-\omega t)}$$

The effective impedance looking into the line at $z = 0$

$$Z_0 = \frac{v(0,t)}{i(0,t)} = \sqrt{\frac{L_l}{C_l}}$$

is called the *characteristic impedance* of the transmission line. For a lossless transmission line this is purely resistive.

If we cut the transmission line at an arbitrary length and connect it to a load with this impedance there is no reflected wave and the source impedance is unaltered. The load resistor is said to be *matched*.

If on the other hand the load impedance is different there will be a reflected wave and there will be a partial standing wave. There will be a modulation of the voltage amplitude.

Power will be reflected and the power *reflection coefficient* is very similar to that for a plane wave impinging on an interface:

$$\mathcal{R} = \left| \frac{Z_{load} - Z_0}{Z_{load} + Z_0} \right|^2$$

The source impedance is also altered:

$$Z_{source} = Z_0 \frac{Z_{load} - iZ_0 \tan(kl)}{Z_0 - iZ_{load} \tan(kl)}$$

Note that this is periodic in the length of the transmission line. This means that if one has bad receiving conditions for the TV, one might improve the TV signal by changing the length of the TV-cable

PROPAGATION OF WAVES BETWEEN CONDUCTING PLANES

I will here just give a brief summary of important results. The separation between the planes is b. There will be standing wave solution in the perpendicular direction and plane wave motion in the parallel direction.

$$\sim \sin(k_c y)e^{i(k_g z - \omega t)}$$

where

$$k_c = n\pi/b \quad ; \quad n = 1, 2, 3\ldots$$

The integer n characterizes the mode and is termed the mode number.

Let

$$\omega = ck_0$$

Then

$$\boxed{k_0^2 = k_c^2 + k_g^2}$$

or

$$\boxed{\frac{1}{\lambda_0^2} = \frac{1}{\lambda_c^2} + \frac{1}{\lambda_g^2}}$$

For a given mode number the phase velocity is

$$u_{ph} = \omega/k_g = ck_0/k_g = c\sqrt{1 + \left(k_c^2/k_g^2\right)}$$

and the group velocity

$$u_{gr} = d\omega/dk_g = cdk_0/dk_g = c\Big/\sqrt{1 + \left(k_c^2/k_g^2\right)}$$

and

$$u_{ph}u_{gr} = c^2$$

There is of course no propagation for a mode if

$$k_g^2 \leq 0$$

which means that

$$k_0 > k_c$$

must be satisfied, or expressed in another way

$$\omega > \omega_c \quad ; \quad \omega_c = ck_c = nc\pi/b$$

the frequency has to exceed the *cutoff frequency*. The corresponding wavelength,

$$\lambda_c = 2b/n,$$

is called the *cutoff wavelength*.

WAVES IN HOLLOW CONDUCTORS

There are no longer *TEM* waves. There are *TE* and *TM* modes. For the *TE* mode the electric field component is transverse, i.e., pointing perpendicular to the propagation vector $\mathbf{k}_g$. The magnetic component is partly pointing in the direction of propagation. For the *TM* modes the magnetic component is transverse and the electric is not.

The equations to be solved are

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{bmatrix}\mathbf{E}\\ \mathbf{B}\end{bmatrix} = 0$$

and we seek solutions of the form

$$\begin{bmatrix}\mathbf{E}\\ \mathbf{B}\end{bmatrix} = \begin{bmatrix}\mathbf{E}_0(x,y)\\ \mathbf{B}_0(x,y)\end{bmatrix} e^{i(k_g z - \omega t)}$$

Substituting this ansatz into the differential equation gives

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k_g^2 + \frac{\omega^2}{c^2}\right)\begin{bmatrix}\mathbf{E}_0\\ \mathbf{B}_0\end{bmatrix} e^{i(k_g z - \omega t)} = 0$$

and

$$\left(\nabla_t^2 + k_c^2\right)\begin{bmatrix}\mathbf{E}_0\\ \mathbf{B}_0\end{bmatrix} = 0$$

where

$$\nabla_t^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and

$$k_c^2 = k_0^2 - k_g^2$$

The solutions to this differential equation for a rectangular wave guide are *TE* and *TM* modes. These modes have the following cutoff frequencies

$$\omega_{mn} = \frac{ck_c}{\sqrt{\mu\varepsilon}} = \frac{\pi c}{\sqrt{\mu\varepsilon}}\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

where we have made the wave guide more general, filled with a dielectric. Let $a>b$.

In most cases only the lowest mode is most important. For *TE* modes at least one of the integers must be non zero. For the *TM* modes both must be non-zero. This means that the TE_{10} mode is the one with lowest cut off frequency and hence of greatest importance.

OPTICAL FIBERS

In the hollow conductor wave guides the solutions can be viewed as resulting from multiple reflections of waves at the conducting boundaries.

In the case of optical fibres we no longer have an empty space (or a space filled with a dielectric) surrounded by metallic walls. Instead we have a dielectric medium surrounded by walls of a dielectric of lower optical density (lower value of refractive index). If the angle of incidence exceeds the critical angle

$$\theta_c \equiv \sin^{-1}\left(n_2/n_1\right)$$

there is total reflection also in this case. In principle we have the same behavior as in the case of a hollow conductor, but there are differences:

1) There is a minimum angle of incidence, which affects the cut-off wavelength or frequency.

2) The reflected wave excites evanescent fields outside the dielectric.

3) The reflections have a phase shift that depends on the angle of incidence.

Since there are evanescent waves outside the core of the optical fiber, and these can easily lead to leakage due to scratches or other imperfections, one surrounds the core with a protecting layer of dielectric known as cladding.