REFLECTION AND REFRACTION

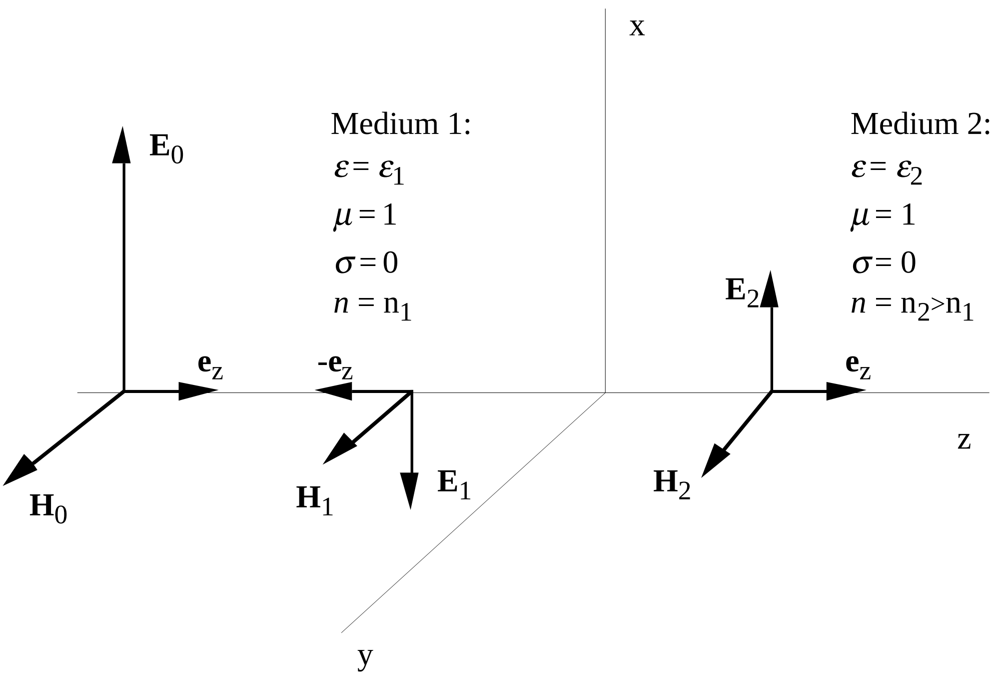
REFLECTION AND TRANSMISSION FOR NORMAL INCIDENCE ON A DIELECTRIC MEDIUM

Assumptions:
Non-magnetic media which means that **B** = **H.**
No damping, purely dielectric media.
No free surface charges.
No free surface current charges.
Flat interfaces with unlimited extent.

Boundary conditions:

E: Tangential component continuous.
D: Normal component continuous
B: Normal component continuous.
H: Tangential component continuous.

Subscript 0, 1 and 2 represent incident, reflected and transmitted wave, respectively.



$$\mathbf{E}_0 = \mathbf{e}_x E_0^0 e^{i(k_1 z - \omega t)}$$
$$\mathbf{E}_1 = -\mathbf{e}_x E_1^0 e^{i(-k_1 z - \omega t)}$$
$$\mathbf{E}_2 = \mathbf{e}_x E_2^0 e^{i(k_2 z - \omega t)}$$

The scalar amplitudes E_i^0 are time-independent and may be complex valued, allowing for a phase difference between the waves.

$$k_1 = n_1 \frac{\omega}{c} = \sqrt{\varepsilon_1}\,\frac{\omega}{c}$$
$$k_2 = n_2 \frac{\omega}{c} = \sqrt{\varepsilon_2}\,\frac{\omega}{c}$$

Since we have assumed non-magnetic materials we have

$$\mathbf{H} \to \mathbf{B} = n\hat{\mathbf{k}} \times \mathbf{E}$$

and

$$\mathbf{H}_0 = \mathbf{e}_y n_1 E_0^0 e^{i(k_1 z - \omega t)}$$
$$\mathbf{H}_1 = \mathbf{e}_y n_1 E_1^0 e^{i(-k_1 z - \omega t)}$$
$$\mathbf{H}_2 = \mathbf{e}_y n_2 E_2^0 e^{i(k_2 z - \omega t)}$$

The boundary conditions for tangential components give

$$E_0^0 - E_1^0 = E_2^0$$

and

$$H_0^0 + H_1^0 = H_2^0$$

or

$$n_1\left(E_0^0 + E_1^0\right) = n_2 E_2^0$$

This gives

$$E_1^0 = \frac{n_2 - n_1}{n_2 + n_1} E_0^0 \quad ;$$

$$E_2^0 = \frac{2n_1}{n_2 + n_1} E_0^0$$

$$H_1^0 = n_1 E_1^0 = \frac{n_2 - n_1}{n_2 + n_1} n_1 E_0^0 = \frac{n_2 - n_1}{n_2 + n_1} H_0^0 \quad ;$$

$$H_2^0 = n_2 E_2^0 = \frac{2n_1}{n_2 + n_1} n_2 E_0^0 = \frac{2n_2}{n_2 + n_1} H_0^0$$

We see that in the case we have here there is a phase change of π for the electric component of the reflected wave but not for the magnetic component and not for the transmitted wave. If instead medium *2* is optically thinner than medium *1* there is a phase change of π for the magnetic component of the reflected wave but not for the electric component. Thus one has to specify if the phase of the wave is represented by the electric or magnetic component.

The average energy flux in the incident wave is

$$\langle S_0 \rangle = \frac{c}{8\pi} \mathrm{Re}\left(\mathbf{E}_0 \times \mathbf{H}_0^* \right)$$

and the *power reflection coefficient* which is defined as the relative amount of energy that is reflected at the boundary:

$$R \equiv \frac{\langle S_1 \rangle \cdot (-\mathbf{e}_z)}{\langle S_0 \rangle \cdot \mathbf{e}_z} = \frac{\left| \mathbf{E}_1 \times \mathbf{H}_1^* \right|}{\left| \mathbf{E}_0 \times \mathbf{H}_0^* \right|} = \frac{\left| E_1^0 \right|^2}{\left| E_0^0 \right|^2}$$

$$\boxed{R = \left(\frac{n_2 - n_1}{n_2 + n_1} \right)^2}$$

The *power transmission coefficient* is defined by

$$T \equiv \frac{\langle S_2 \rangle \cdot \mathbf{e}_z}{\langle S_0 \rangle \cdot \mathbf{e}_z} = \frac{\left|\mathbf{E}_2 \times \mathbf{H}_2^*\right|}{\left|\mathbf{E}_0 \times \mathbf{H}_0^*\right|} = \frac{n_2}{n_1} \frac{\left|E_2^0\right|^2}{\left|E_0^0\right|^2}$$

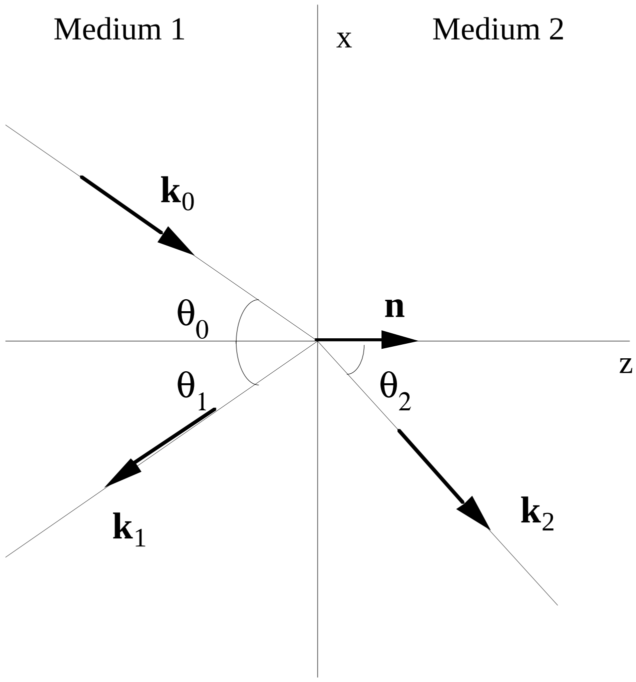
so that

$$\boxed{T = \frac{n_2}{n_1}\left(\frac{2n_1}{n_2 + n_1}\right)^2 = \frac{4n_1 n_2}{\left(n_2 + n_1\right)^2}}$$

Energy conservation means that

$$R + T = 1$$

OBLIQUE INCIDENCE-THE FRESNEL EQUATIONS



$$\mathbf{E}_0 = \mathbf{E}_0^0 e^{i(\mathbf{k}_0\cdot\mathbf{r}-\omega t)}$$

$$\mathbf{H}_0 = \frac{n_1}{k_0}\mathbf{k}_0 \times \mathbf{E}_0$$

$$\mathbf{E}_1 = \mathbf{E}_1^0 e^{i(\mathbf{k}_1\cdot\mathbf{r}-\omega t)}$$

$$\mathbf{H}_1 = \frac{n_1}{k_1}\mathbf{k}_1 \times \mathbf{E}_1$$

$$\mathbf{E}_2 = \mathbf{E}_2^0 e^{i(\mathbf{k}_2\cdot\mathbf{r}-\omega t)}$$

$$\mathbf{H}_2 = \frac{n_2}{k_2}\mathbf{k}_2 \times \mathbf{E}_2$$

Note that we use the **H** fields instead of the fundamental **B** fields because the boundary conditions for these fields are the same as for the **E** fields.

Now, the tangential components of the fields have to be continuous across the boundary. For this to be possible the periodicities of the field vectors have to be equal at the boundary:

$$\mathbf{k}_0 \cdot \mathbf{e}_x = \mathbf{k}_1 \cdot \mathbf{e}_x = \mathbf{k}_2 \cdot \mathbf{e}_x$$

For symmetry reasons all three propagation vectors are coplanar, i.e. are in the same plane, the *plane of incidence* (The surface normal **n** is also in this plane).

This means that

$$k_0 \sin\theta_0 = k_1 \sin\theta_1 = k_2 \sin\theta_2$$

This can also be viewed as conservation of the momentum parallel to interface. This conservation holds if the interface is smooth. A rough interface or an intentionally created periodic structure at the interface can relax this condition. Momentum parallel to the interface can then be absorbed from, or provided to, the reflected and refracted waves.

Now,

$$k_0 = k_1 \Rightarrow \boxed{\theta_0 = \theta_1}$$

and

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}$$ *Snell's law*

Now we have determined the propagation direction of the reflected and refracted waves. The amplitudes of the field components are obtained from using the boundary conditions. It is enough to use the boundary conditions for the tangential components of the **E** and **H** fields. The conditions for the normal components of the **B** and **D** fields give no more information.

The boundary conditions are:

$$\left(\mathbf{E}_0 + \mathbf{E}_1\right) \times \mathbf{n} = \mathbf{E}_2 \times \mathbf{n};$$
$$\left(\mathbf{H}_0 + \mathbf{H}_1\right) \times \mathbf{n} = \mathbf{H}_2 \times \mathbf{n}$$

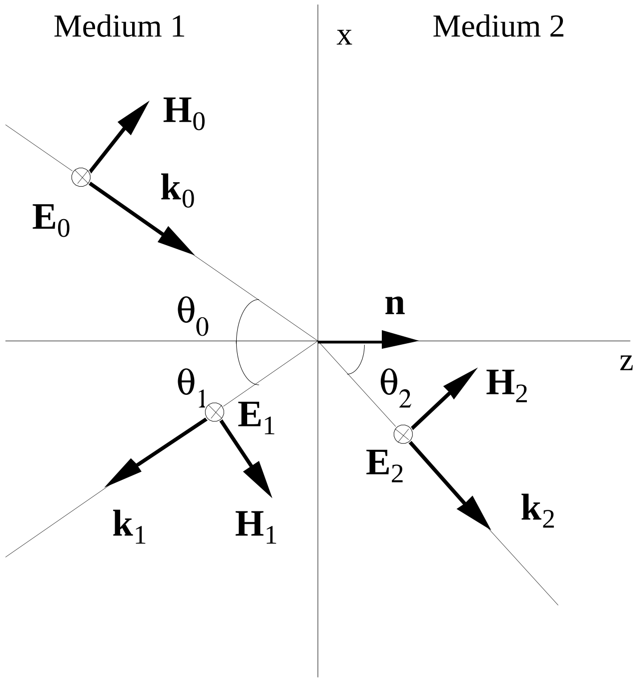
The last can be rewritten in terms of the electric field vectors and we have

$$\left(\mathbf{E}_0 + \mathbf{E}_1\right) \times \mathbf{n} = \mathbf{E}_2 \times \mathbf{n};$$
$$\left(\mathbf{k}_0 \times \mathbf{E}_0 + \mathbf{k}_1 \times \mathbf{E}_1\right) \times \mathbf{n} = \left(\mathbf{k}_2 \times \mathbf{E}_2\right) \times \mathbf{n}$$

Any plane wave impinging on the interface can be written as a linear combination of two waves, one with the electric vector polarized parallel to the plane of incidence (*p-polarized*), and one with the electric vector perpendicular to the plane of incidence (*s-polarized*). We may treat these components separately.

E PERPENDICULAR TO THE PLANE OF INCIDENCE.

S-polarized waves.



All **E** vectors are in the minus *y* direction. This means that the first boundary condition makes:

$$E_0^0 + E_1^0 = E_2^0$$

The second we expand using the triple curl product

$$\mathbf{n}\times\left(\mathbf{k}_0\times\mathbf{E}_0\right)+\mathbf{n}\times\left(\mathbf{k}_1\times\mathbf{E}_1\right)=\mathbf{n}\times\left(\mathbf{k}_2\times\mathbf{E}_2\right)$$

$$\mathbf{k}_0\underbrace{\left(\mathbf{n}\cdot\mathbf{E}_0\right)}_{0}-\mathbf{E}_0\underbrace{\left(\mathbf{n}\cdot\mathbf{k}_0\right)}_{k_0\cos\theta_0}$$

$$+\mathbf{k}_1\underbrace{\left(\mathbf{n}\cdot\mathbf{E}_1\right)}_{0}-\mathbf{E}_1\underbrace{\left(\mathbf{n}\cdot\mathbf{k}_1\right)}_{-k_1\cos\theta_1}=\mathbf{k}_2\underbrace{\left(\mathbf{n}\cdot\mathbf{E}_2\right)}_{0}-\mathbf{E}_2\underbrace{\left(\mathbf{n}\cdot\mathbf{k}_2\right)}_{k_2\cos\theta_2}$$

or

$$\mathbf{E}_0 k_0 \cos\theta_0 - \mathbf{E}_1 k_1 \cos\theta_1 = \mathbf{E}_2 k_2 \cos\theta_2$$

or since

$$k_0 = k_1 \quad ; \quad \theta_1 = \theta_0 \quad ; \quad k_2/k_1 = n_2/n_1$$

we have

$$\left(E_0^0 - E_1^0\right)\cos\theta_0 = \frac{n_2}{n_1} E_2^0 \cos\theta_2$$

Combining the two relations gives

$$\begin{aligned} E_1^0 &= \frac{\cos\theta_0 - (n_2/n_1)\cos\theta_2}{\cos\theta_0 + (n_2/n_1)\cos\theta_2} E_0^0 \\ &= \frac{\sin(\theta_2 - \theta_0)}{\sin(\theta_2 + \theta_0)} E_0^0 \\ E_2^0 &= \frac{2\cos\theta_0}{\cos\theta_0 + (n_2/n_1)\cos\theta_2} E_0^0 \\ &= \frac{2\cos\theta_0 \sin\theta_2}{\sin(\theta_2 + \theta_0)} E_0^0 \end{aligned}$$

Reflection and transmission coefficients

$$R = \frac{\langle \mathbf{S}_1 \rangle \cdot (-\mathbf{n})}{\langle \mathbf{S}_0 \rangle \cdot \mathbf{n}} = \frac{\frac{c}{8\pi}\left|E_1^0\right|^2 n_1 \hat{\mathbf{k}}_1 \cdot (-\mathbf{n})}{\frac{c}{8\pi}\left|E_0^0\right|^2 n_0 \hat{\mathbf{k}}_0 \cdot \mathbf{n}} = \frac{\left|E_1^0\right|^2}{\left|E_0^0\right|^2}$$

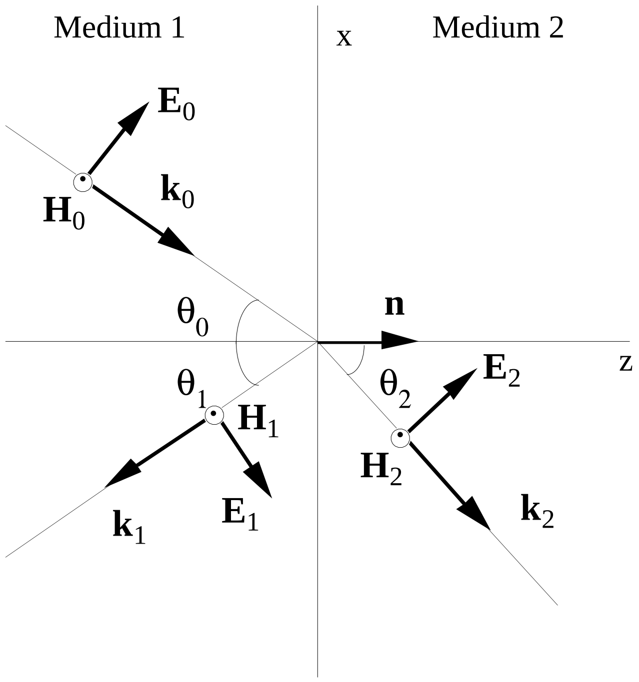
$$T = \frac{\langle \mathbf{S}_2 \rangle \cdot \mathbf{n}}{\langle \mathbf{S}_0 \rangle \cdot \mathbf{n}} = \frac{\frac{c}{8\pi}\left|E_2^0\right|^2 n_2 \hat{\mathbf{k}}_2 \cdot \mathbf{n}}{\frac{c}{8\pi}\left|E_0^0\right|^2 n_0 \hat{\mathbf{k}}_0 \cdot \mathbf{n}} = \frac{\left|E_2^0\right|^2 n_2 \cos\theta_2}{\left|E_0^0\right|^2 n_1 \cos\theta_0}$$

$$R_\perp = \frac{\left|E_1^0\right|^2}{\left|E_0^0\right|^2} = \frac{\sin^2(\theta_2 - \theta_0)}{\sin^2(\theta_2 + \theta_0)}$$

$$\begin{aligned}
T_\perp &= \frac{\left|E_2^0\right|^2 n_2 \cos\theta_2}{\left|E_0^0\right|^2 n_1 \cos\theta_0} \\
&= 4\frac{n_2 \cos\theta_2}{n_1 \cos\theta_0}\frac{\cos^2\theta_0 \sin^2\theta_2}{\sin^2(\theta_2 + \theta_0)} \\
&= 4\frac{\sin\theta_0 \cos\theta_2}{\sin\theta_2 \cos\theta_0}\frac{\cos^2\theta_0 \sin^2\theta_2}{\sin^2(\theta_2 + \theta_0)} \\
&= 4\frac{\sin\theta_0 \cos\theta_2 \cos\theta_0 \sin\theta_2}{\sin^2(\theta_2 + \theta_0)} \\
&= \frac{\sin 2\theta_0 \sin 2\theta_2}{\sin^2(\theta_2 + \theta_0)}
\end{aligned}$$

E PARALLEL TO THE PLANE OF INCIDENCE.

P-polarized waves.



Now all **H** vectors are pointing in the *y* direction, i.e., out of the plane of the figure.

We can make use of our results from the *s*-polarized case. We just change the electric fields to the refractive index times the electric fields in the condition for the fields:

$$E_0^0 + E_1^0 = E_2^0 \Rightarrow n_1\left(E_0^0 + E_1^0\right) = n_2 E_2^0$$

and in the condition for the **H** fields we change the refractive index times the electric fields to the electric fields:

$$\left(E_0^0 - E_1^0\right)\cos\theta_0 = \frac{n_2}{n_1} E_2^0 \cos\theta_2 \Rightarrow \left(E_0^0 - E_1^0\right)\cos\theta_0 = E_2^0 \cos\theta_2$$

from which we obtain

$$
\begin{aligned}
E_1^0 &= \frac{\cos\theta_0 - (n_1/n_2)\cos\theta_2}{\cos\theta_0 + (n_1/n_2)\cos\theta_2} E_0^0 \\
&= \frac{\tan(\theta_2 - \theta_0)}{\tan(\theta_2 + \theta_0)} E_0^0 \\
E_2^0 &= \frac{2\cos\theta_0}{\cos\theta_2 + (n_2/n_1)\cos\theta_0} E_0^0 \\
&= \frac{2\cos\theta_0 \sin\theta_2}{\sin(\theta_0 + \theta_2)\cos(\theta_0 - \theta_2)} E_0^0
\end{aligned}
$$

Reflection and transmission coefficients

$$R_{//} = \frac{\left|E_1^0\right|^2}{\left|E_0^0\right|^2} = \frac{\tan^2(\theta_2 - \theta_0)}{\tan^2(\theta_2 + \theta_0)}$$

$$T_{//} = \frac{\left|E_2^0\right|^2 n_2 \cos\theta_2}{\left|E_0^0\right|^2 n_1 \cos\theta_0}$$

$$= 4\frac{n_2 \cos\theta_2}{n_1 \cos\theta_0} \frac{\cos^2\theta_0 \sin^2\theta_2}{\sin^2(\theta_0 + \theta_2)\cos^2(\theta_0 - \theta_2)}$$

$$= 4\frac{\sin\theta_0 \cos\theta_2}{\sin\theta_2 \cos\theta_0} \frac{\cos^2\theta_0 \sin^2\theta_2}{\sin^2(\theta_0 + \theta_2)\cos^2(\theta_0 - \theta_2)}$$

$$= 4\frac{\sin\theta_0 \cos\theta_0 \sin\theta_2 \cos\theta_2}{\sin^2(\theta_0 + \theta_2)\cos^2(\theta_0 - \theta_2)}$$

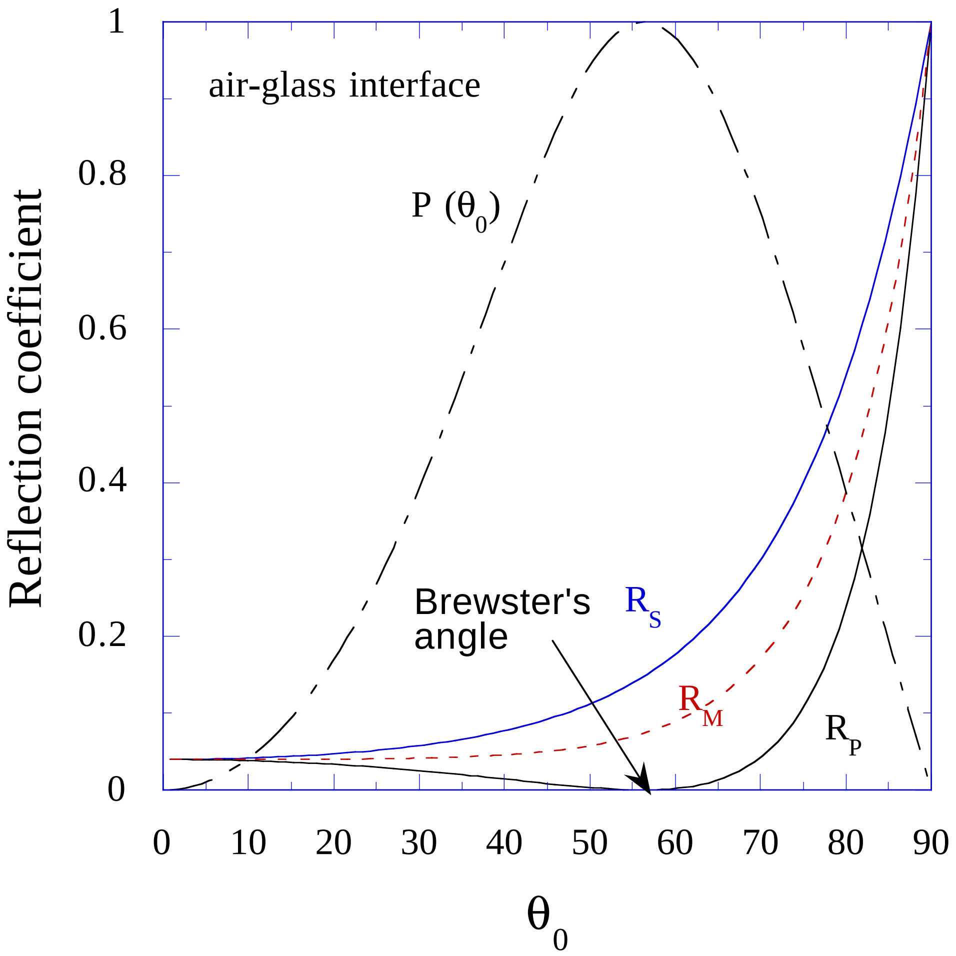
$$= \frac{\sin 2\theta_0 \sin 2\theta_2}{\sin^2(\theta_0 + \theta_2)\cos^2(\theta_0 - \theta_2)}$$

We see that there is no reflection if

$$\theta_2 + \theta_1 = \theta_2 + \theta_0 = \pi/2$$

This happens for the angle of incidence

$$\theta_0 = \theta_B = \tan^{-1}(n_2/n_1)$$ *Brewster's Angle*


air-glass interface
P (θ_0)
Brewster's angle
R_S
R_M
R_P
Reflection coefficient
θ_0
0
0.2
0.4
0.6
0.8
1
0
10
20
30
40
50
60
70
80
90