PLANE WAVES IN CONDUCTING MEDIA

We will now change the book keeping of the charge and current densities to be able to treat metals in a more realistic way.

Let us return to Maxwell's equations:

$$\nabla\cdot\mathbf{D}=4\pi\rho_f=4\pi\rho_{ext}+4\pi\rho_c$$
$$\nabla\cdot\mathbf{B}=0$$
$$\nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}$$
$$\nabla\times\mathbf{H}=\frac{4\pi}{c}\mathbf{J}_f+\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}=\frac{4\pi}{c}\mathbf{J}_{ext}+\frac{4\pi}{c}\mathbf{J}_c+\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}$$

where we have divided the densities in external and those due to the conduction band electrons.

Now we express the equations in terms of the true fields and add magnetization currents to the conduction currents. The magnetization currents have both spin (paramagnetic effects) and orbital parts (diamagnetic effects). Neither of these are destroyed by electron collisions. These currents have very small effects.

$$\nabla\cdot\mathbf{E}=4\pi\rho_t=4\pi\rho_{ext}+4\pi\rho_b+4\pi\rho_c$$
$$\nabla\cdot\mathbf{B}=0$$
$$\nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}$$
$$\begin{aligned}\nabla\times\mathbf{B}&=4\pi\nabla\times\mathbf{M}_b+4\pi\nabla\times\mathbf{M}_c+\frac{4\pi}{c}\left[\mathbf{J}_{ext}+\mathbf{J}_c\right]+\frac{4\pi}{c}\frac{\partial\mathbf{P}_b}{\partial t}+\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\\&=4\pi\nabla\times\mathbf{M}_b+4\pi\nabla\times\mathbf{M}_c+\frac{4\pi}{c}\mathbf{J}_{ext}+\frac{4\pi}{c}\frac{\partial\mathbf{P}_b}{\partial t}+\frac{4\pi}{c}\frac{\partial\mathbf{P}_c}{\partial t}+\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\end{aligned}$$

We introduce new auxiliary fields, $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{H}}$,

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}_b = \varepsilon\mathbf{E}; \;\; \tilde{\mathbf{D}} = \mathbf{E} + 4\pi\mathbf{P}_b + 4\pi\mathbf{P}_c = \varepsilon\mathbf{E} + 4\pi\mathbf{P}_c = \tilde{\varepsilon}\mathbf{E}$$
$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}_b = \mathbf{B}/\mu; \;\; \tilde{\mathbf{H}} = \mathbf{B} - 4\pi\mathbf{M}_b - 4\pi\mathbf{M}_c = \mathbf{B}/\mu - 4\pi\mathbf{M}_c = \mathbf{B}/\tilde{\mu}$$

and we may write

$$\nabla \cdot \tilde{\mathbf{D}} = 4\pi\rho_{ext}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \tilde{\mathbf{H}} = \frac{4\pi}{c}\mathbf{J}_{ext} + \frac{1}{c}\frac{\partial \tilde{\mathbf{D}}}{\partial t}$$

Let us now change into the Fourier transformed version and make use of the constitutive relations:

$$i\tilde{\varepsilon}(\mathbf{q},\omega)\mathbf{q} \cdot \mathbf{E}(\mathbf{q},\omega) = 4\pi\rho_{ext}(\mathbf{q},\omega)$$
$$i\tilde{\mu}(\mathbf{q},\omega)\mathbf{q} \cdot \tilde{\mathbf{H}}(\mathbf{q},\omega) = 0$$
$$i\mathbf{q} \times \mathbf{E} = i\omega\frac{1}{c}\tilde{\mu}(\mathbf{q},\omega)\tilde{\mathbf{H}}(\mathbf{q},\omega)$$
$$i\mathbf{q} \times \tilde{\mathbf{H}} = \frac{4\pi}{c}\mathbf{J}_{ext}(\mathbf{q},\omega) - i\omega\tilde{\varepsilon}(\mathbf{q},\omega)\frac{1}{c}\mathbf{E}(\mathbf{q},\omega)$$

Let us now determine the modified, or extended, dielectric function. The equation of continuity is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

This equation is valid for the total densities, or for the separate parts of the densities. Let $\mathbf{J}_c$ from now on denote the polarization current. Then in particular:

$$\frac{\partial \rho_c}{\partial t} + \nabla \cdot \mathbf{J}_c = 0$$

The Fourier transformed version is

$$-i\omega\rho_c(\mathbf{q},\omega) = -i\mathbf{q}\cdot\mathbf{J}_c(\mathbf{q},\omega)$$

We are also helped by Ohm's law:

$$\mathbf{J}_c(\mathbf{q},\omega) = \sigma(\mathbf{q},\omega)\mathbf{E}(\mathbf{q},\omega)$$

and thus

$$\rho_c(\mathbf{q},\omega) = \frac{1}{\omega}\mathbf{q}\cdot\mathbf{J}_c(\mathbf{q},\omega) = \frac{\sigma(\mathbf{q},\omega)}{\omega}\mathbf{q}\cdot\mathbf{E}(\mathbf{q},\omega)$$

We may eliminate the charge and current density due to the conduction band electrons from Maxwell's equations

$$\begin{aligned}
&i\varepsilon(\mathbf{q},\omega)\mathbf{q}\cdot\mathbf{E}(\mathbf{q},\omega) = 4\pi\rho_{ext}(\mathbf{q},\omega) + \frac{4\pi\sigma(\mathbf{q},\omega)}{\omega}\mathbf{q}\cdot\mathbf{E}(\mathbf{q},\omega)\\
&i\tilde{\mu}(\mathbf{q},\omega)\mathbf{q}\cdot\tilde{\mathbf{H}}(\mathbf{q},\omega) = 0\\
&i\mathbf{q}\times\mathbf{E} = i\omega\frac{1}{c}\tilde{\mu}(\mathbf{q},\omega)\tilde{\mathbf{H}}(\mathbf{q},\omega)\\
&i\mathbf{q}\times\tilde{\mathbf{H}} = \frac{4\pi\sigma(\mathbf{q},\omega)}{c}\mathbf{E}(\mathbf{q},\omega) + \frac{4\pi}{c}\mathbf{J}_{ext}(\mathbf{q},\omega) - i\omega\varepsilon(\mathbf{q},\omega)\frac{1}{c}\mathbf{E}(\mathbf{q},\omega)
\end{aligned}$$

and identify

$$\tilde{\varepsilon}(\mathbf{q},\omega) = \varepsilon(\mathbf{q},\omega) + i4\pi\sigma(\mathbf{q},\omega)/\omega$$

Thus we may write Maxwell's equations in the form:

$$\nabla \cdot \tilde{\mathbf{D}} = 4\pi\rho_{ext}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \tilde{\mathbf{H}} = \frac{4\pi}{c}\mathbf{J}_{ext} + \frac{1}{c}\frac{\partial \tilde{\mathbf{D}}}{\partial t}$$

where now the constitutive relations are

$$\tilde{\mathbf{D}}(\mathbf{q},\omega) = \tilde{\varepsilon}(\mathbf{q},\omega)\mathbf{E}(\mathbf{q},\omega);\ \mathbf{B}(\mathbf{q},\omega) = \tilde{\mu}(\mathbf{q},\omega)\tilde{\mathbf{H}}(\mathbf{q},\omega)$$

Let us stay with the Fourier transformed versions of Maxwell's equations in absence of external charge and current densities, operate with $\mathbf{q}\times$ on the two last equations and use the relation

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

We have the following starting equations where we have eliminated an i in all equations

$$\tilde{\varepsilon}(\mathbf{q},\omega)\mathbf{q} \cdot \mathbf{E}(\mathbf{q},\omega) = 0$$
$$\tilde{\mu}(\mathbf{q},\omega)\mathbf{q} \cdot \tilde{\mathbf{H}}(\mathbf{q},\omega) = 0$$
$$\mathbf{q} \times \mathbf{E} = \omega\frac{1}{c}\tilde{\mu}(\mathbf{q},\omega)\tilde{\mathbf{H}}(\mathbf{q},\omega)$$
$$\mathbf{q} \times \tilde{\mathbf{H}} = -\omega\tilde{\varepsilon}(\mathbf{q},\omega)\frac{1}{c}\mathbf{E}(\mathbf{q},\omega)$$

If we first assume that the dielectric function and magnetic permeability are finite, non-zero we have (then the **E** and **H** fields are purely transverse):

$$
\begin{aligned}
-\mathbf{E}(\mathbf{q}\cdot\mathbf{q}) &= \mathbf{q}(\mathbf{q}\cdot\mathbf{E}) - \mathbf{E}(\mathbf{q}\cdot\mathbf{q}) \\
&= \underset{\leftarrow}{} \left[\mathbf{q}\times(\mathbf{q}\times\mathbf{E}) = \omega\frac{1}{c}\tilde{\mu}(\mathbf{q},\omega)\mathbf{q}\times\tilde{\mathbf{H}}(\mathbf{q},\omega)\right]_{\rightarrow} \\
&= \omega\frac{1}{c}\tilde{\mu}(\mathbf{q},\omega)\left[-\omega\tilde{\varepsilon}(\mathbf{q},\omega)\frac{1}{c}\mathbf{E}(\mathbf{q},\omega)\right]
\end{aligned}
$$

or

$$
(i\mathbf{q})^2\,\mathbf{E} - (i\omega)^2\frac{\tilde{\mu}\varepsilon}{c^2}\mathbf{E} - i\omega\frac{4\pi\tilde{\mu}\sigma}{c^2}\mathbf{E} = 0
$$

The other equation gives

$$
\begin{aligned}
-\tilde{\mathbf{H}}(\mathbf{q}\cdot\mathbf{q}) &= \mathbf{q}(\mathbf{q}\cdot\tilde{\mathbf{H}}) - \tilde{\mathbf{H}}(\mathbf{q}\cdot\mathbf{q}) \\
&= \underset{\leftarrow}{} \left[\mathbf{q}\times(\mathbf{q}\times\tilde{\mathbf{H}}) = -\omega\tilde{\varepsilon}(\mathbf{q},\omega)\frac{1}{c}\mathbf{q}\times\mathbf{E}(\mathbf{q},\omega)\right]_{\rightarrow} \\
&= -\omega\tilde{\varepsilon}(\mathbf{q},\omega)\frac{1}{c}\left[\omega\frac{1}{c}\tilde{\mu}(\mathbf{q},\omega)\tilde{\mathbf{H}}(\mathbf{q},\omega)\right]
\end{aligned}
$$

or

$$
(i\mathbf{q})^2\,\tilde{\mathbf{H}} - (i\omega)^2\frac{\tilde{\mu}\varepsilon}{c^2}\tilde{\mathbf{H}} - i\omega\frac{4\pi\tilde{\mu}\sigma}{c^2}\tilde{\mathbf{H}} = 0
$$

Thus for a specific Fourier component we have the two differential equations:

$$
\boxed{\nabla^2\mathbf{E} - \frac{\varepsilon\tilde{\mu}}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} - \frac{4\pi\sigma\tilde{\mu}}{c^2}\frac{\partial\mathbf{E}}{\partial t} = 0}
$$

and

$$
\boxed{\nabla^2\tilde{\mathbf{H}} - \frac{\varepsilon\tilde{\mu}}{c^2}\frac{\partial^2\tilde{\mathbf{H}}}{\partial t^2} - \frac{4\pi\sigma\tilde{\mu}}{c^2}\frac{\partial\tilde{\mathbf{H}}}{\partial t} = 0}
$$

or

$$\nabla^2 \mathbf{E} - \frac{\tilde{\varepsilon}\tilde{\mu}}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

and

$$\nabla^2 \tilde{\mathbf{H}} - \frac{\tilde{\varepsilon}\tilde{\mu}}{c^2}\frac{\partial^2 \tilde{\mathbf{H}}}{\partial t^2} = 0$$

Discussion: We have at several occasions mentioned that the simple constitutive relations are not valid in general. We should have convolution integrals instead. However, if we have a specific Fourier component of the field with a specific frequency and momentum the Maxwell equations are valid and the dielectric function, magnetic permeability and conductivity should be the Fourier transforms at those specific frequencies and momenta.

Before we continue with the solutions to these two wave equations let us briefly return to the first two Maxwell's equations:

$$\tilde{\varepsilon}(\mathbf{q},\omega)\mathbf{q}\cdot\mathbf{E}(\mathbf{q},\omega) = 0$$
$$\tilde{\mu}(\mathbf{q},\omega)\mathbf{q}\cdot\tilde{\mathbf{H}}(\mathbf{q},\omega) = 0$$

These have solutions, not mentioned in the text book. These are *longitudinal solutions* and they are found by putting $\tilde{\varepsilon}(\mathbf{q},\omega) = 0$ or $\tilde{\mu}(\mathbf{q},\omega) = 0$, respectively.

The first type gives plasmons, longitudinal phonons or some other longitudinal excitation, all dependent on the system we are considering. The other gives magnetic excitations like magnons or spin waves.

Let us now return to our transverse solutions. The wave equations are valid for solutions that are plane waves that vary harmonically in space and time. The last equation is also valid for the magnetic inductions. We may just multiply all terms in the equation with the magnetic permeability and find the identical equation for the **B** field.

Let

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \quad ;$$
$$\mathbf{B} = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

Put this into the equations and find

$$\left(k^2 - \frac{\tilde{\varepsilon}\tilde{\mu}\omega^2}{c^2}\right)\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = 0 \;;$$
$$\left(k^2 - \frac{\tilde{\varepsilon}\tilde{\mu}\omega^2}{c^2}\right)\mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = 0$$

and from this

$$k^2 = \frac{\tilde{\varepsilon}\tilde{\mu}\omega^2}{c^2} = \frac{\tilde{\mu}\omega^2}{c^2}\left(\varepsilon + i\frac{4\pi\sigma}{\omega}\right) = \frac{\varepsilon\tilde{\mu}\omega^2}{c^2}\left(1 + i\frac{4\pi\sigma}{\varepsilon\omega}\right)$$

Thus the propagation constant, the wave number, k, becomes complex valued:

$$k \equiv \alpha + i\beta$$

and the spatial variation becomes:

$$e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikr\cos\theta} = e^{ik\zeta} = e^{-\beta\zeta}e^{i\alpha\zeta}$$

The wave decreases in amplitude as it propagates and the wave is damped.

Assuming that ε and σ are real valued, which they in general are not, we get the relations for α and β given in equation (5.75).

We may write the propagation constant as

$$k = \tilde{n}\frac{\omega}{c} \;;\; \tilde{n} \equiv \sqrt{\tilde{\varepsilon}\tilde{\mu}} \,, \qquad \textit{complex index of refraction}$$

Now,

$$\mathbf{B}_0 = \tilde{n}\hat{\mathbf{k}} \times \mathbf{E}_0$$

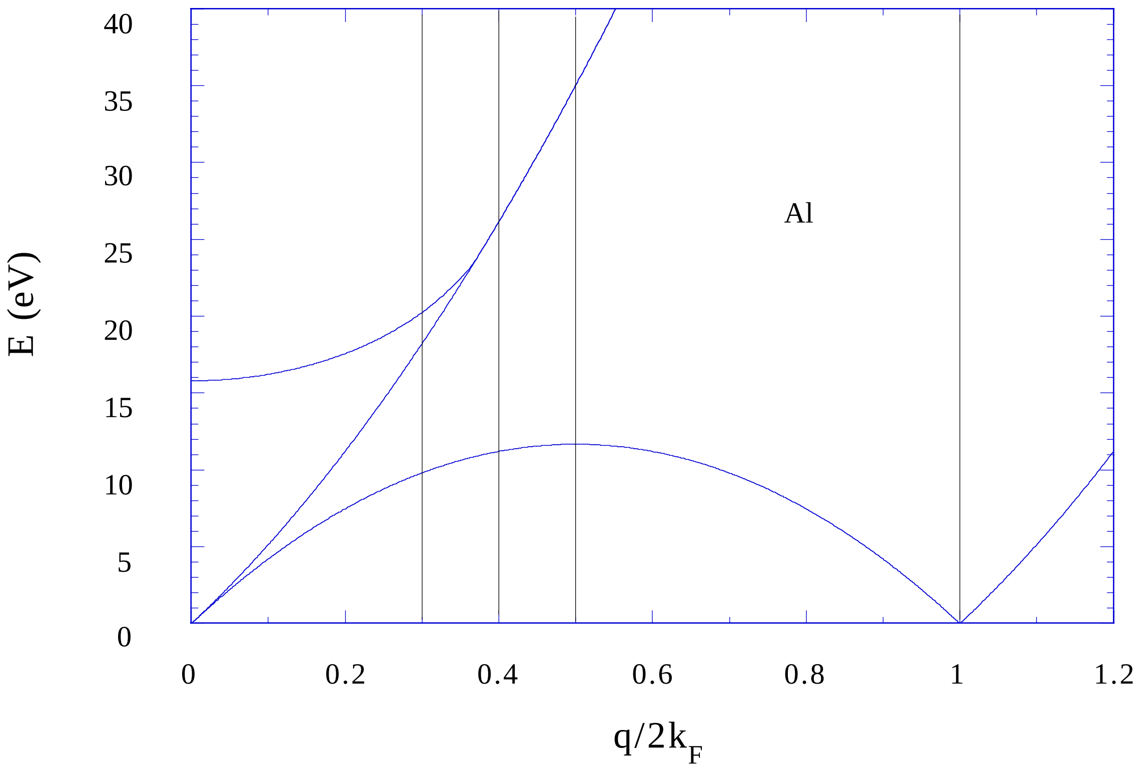
If we have a complex refractive index the electric and magnetic fields are no longer in phase. If the refractive index is large the magnetic induction is much larger than the electric field:

$$B_0 = |\tilde{n}| E_0$$

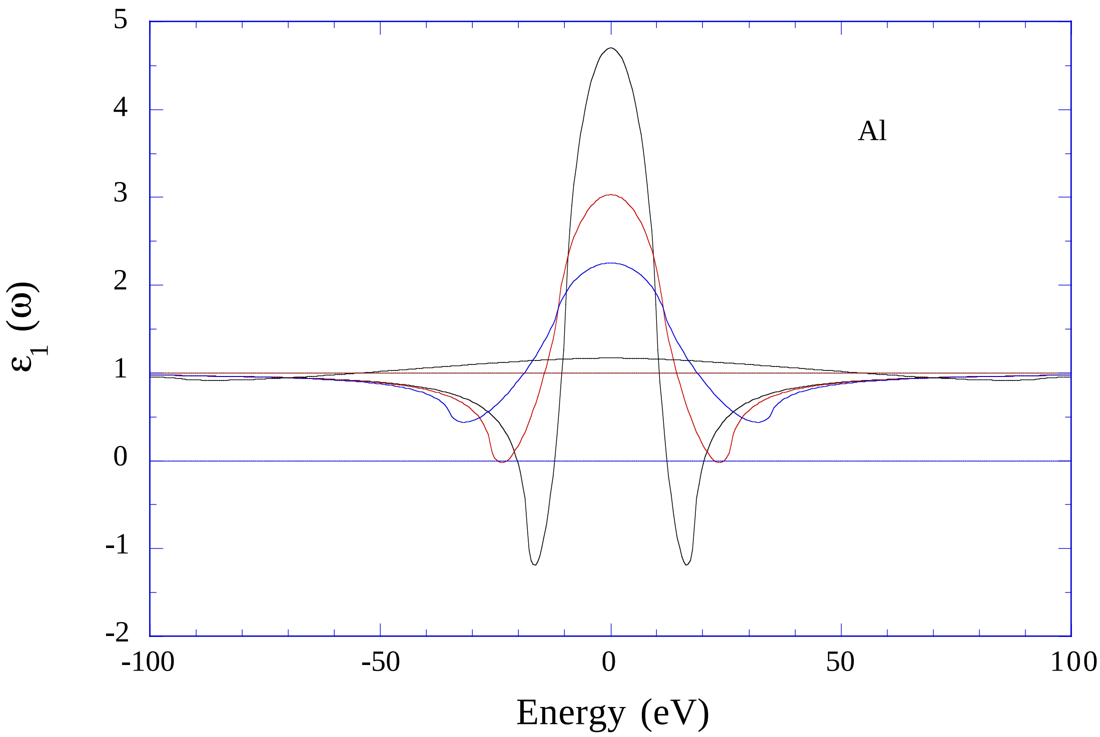
Will this mean that the energy density is dominated by the energy in the magnetic part of the field?

$$\begin{aligned}
\langle \mathscr{E} \rangle &= \frac{1}{16\pi}\left(\mathbf{E}_0 \cdot \tilde{\mathbf{D}}_0^* + \tilde{\mathbf{H}}_0 \cdot \mathbf{B}_0^*\right) \\
&= \frac{1}{16\pi}\left(\tilde{\varepsilon}^* E_0^2 + \tilde{\mu}^* \tilde{H}_0^2\right) \\
&= \frac{1}{16\pi}\left(\tilde{\varepsilon}^* E_0^2 + \frac{1}{\tilde{\mu}} B_0^2\right) \\
&= \frac{1}{16\pi}\left(\tilde{\varepsilon}^* E_0^2 + \frac{\tilde{\varepsilon}\tilde{\mu}}{\tilde{\mu}} E_0^2\right) \\
&= \frac{1}{8\pi}\mathrm{Re}\left(\tilde{\varepsilon}\right) E_0^2
\end{aligned}$$

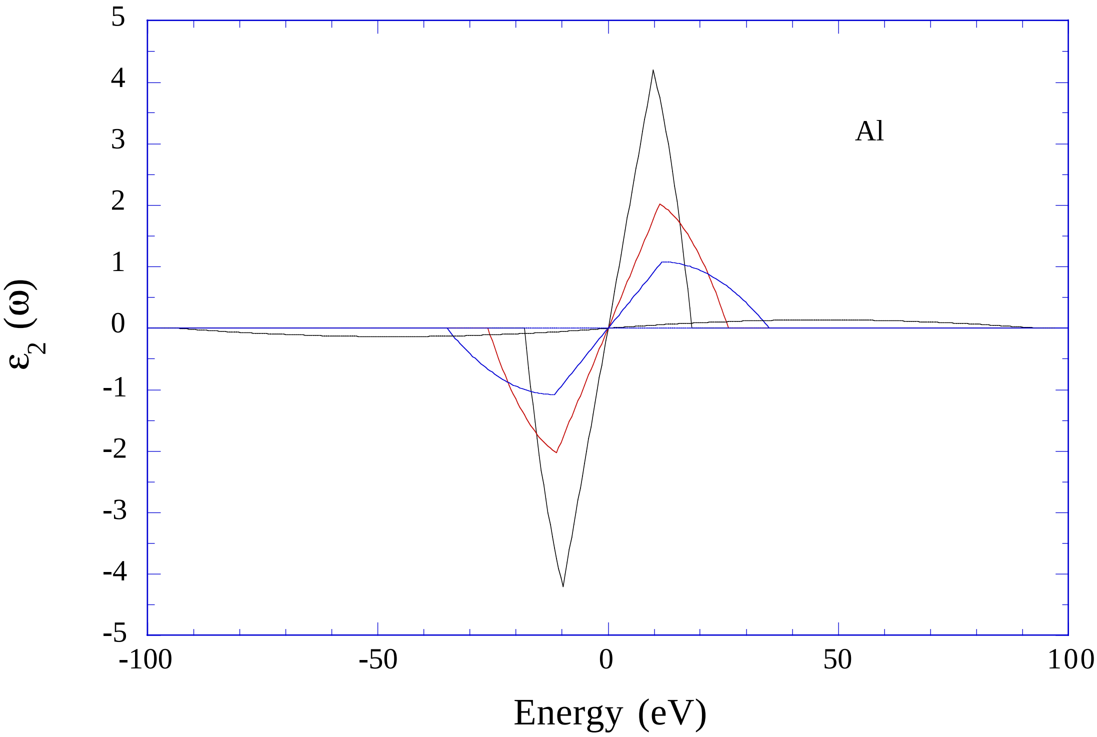
Both type of fields still give the same contribution to the energy density. Note that we have here included the energy stored in the conduction band electrons. If we had not, the dielectric function appearing in the first term would have been the one without the tilde, i.e., the one without contributions from the conduction-band electrons. It is all a matter of book keeping. In the text book one has chosen not to include this energy contribution in the energy density of the field and hence find that the magnetic part dominates in a metal at low frequencies. They have a different definition of the **D** vector than we have here.



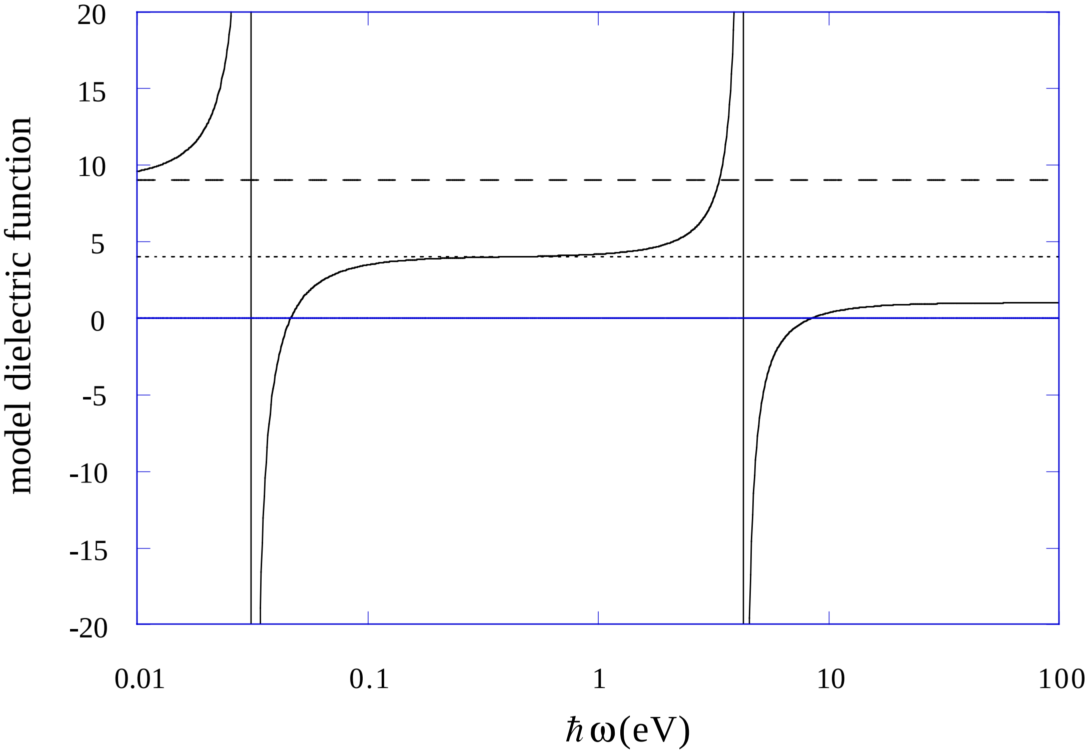
Excitation regions in the ωq-plane for Aluminum



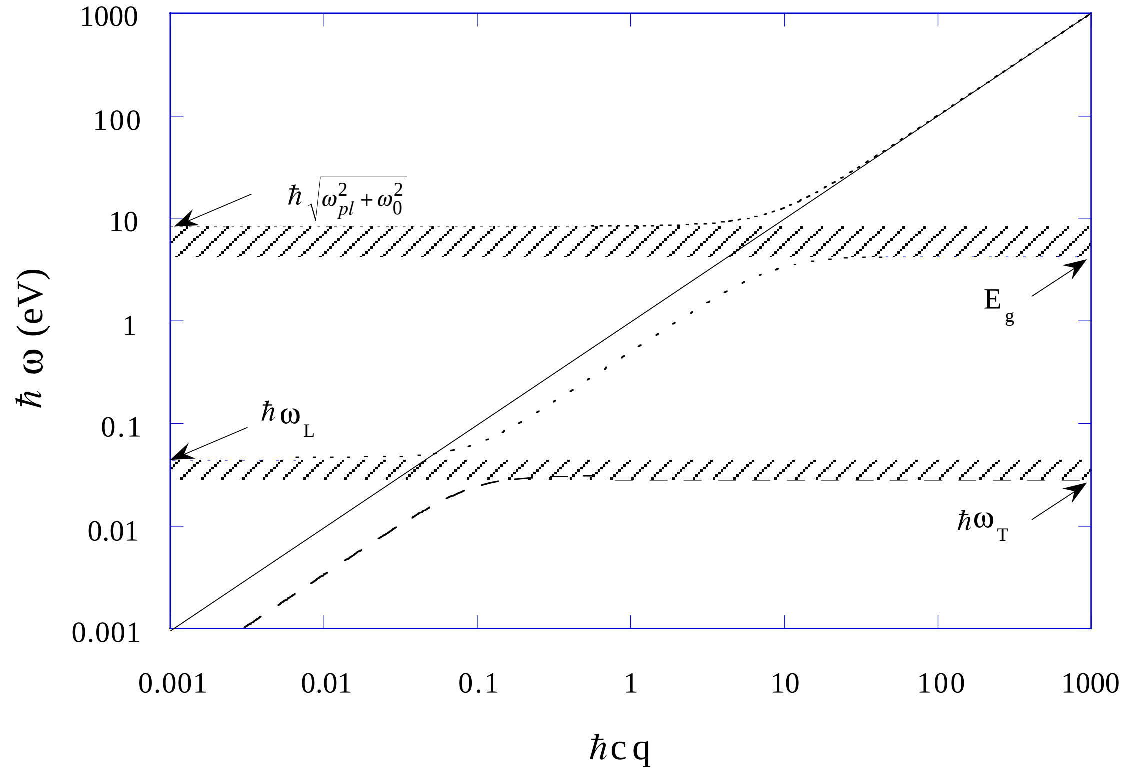
Real part of dielectric function along the lines indicated in previous figure.



Imaginary part along the same lines



The dielectric function of a model semiconductor



The transverse modes in such a system