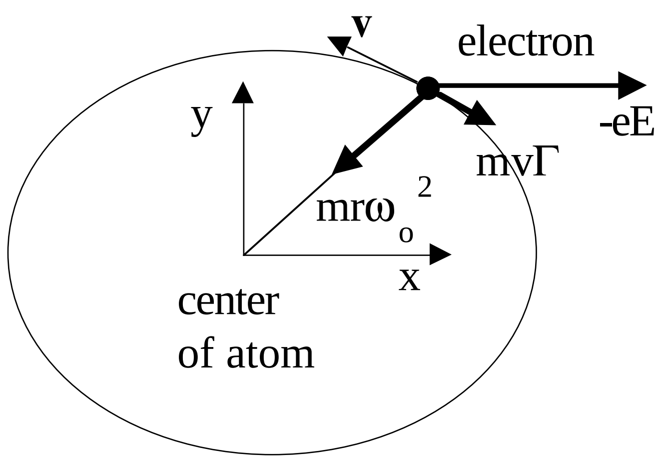
CLASSICAL ELECTRON THEORY

Lorentz' classical model for the dielectric function of insulators

In this model the electrons are assumed to be bound to the nucleus with forces obeying Hooke's law. The forces are assumed to be isotropic and damping can be included through frictional forces proportional to the electron velocity.



Schematic illustration of the electron orbit around the center of the atom. Indicated by thick arrows are the forces acting on the electron: the centripetal force towards the center; the frictional force in the direction opposite to that of the velocity **v**; the external force in the direction opposite to that of the external electric field **E**.

The equation of motion for an electron is

$$m\ddot{\mathbf{r}} + m\Gamma\dot{\mathbf{r}} + m\omega_0^2\mathbf{r} = -e\mathbf{E}$$

This is just the Newton's second law, stating that mass times acceleration equals the force exerted on the particle; we have collected all terms except the external force on the left hand side of the equation. The first term is just the mass times the acceleration; the second the frictional force; the third the restoring force binding the electron to the nucleus, placed at the origin. The term on the right hand side of the equation is the force due to the external field, **E**.

Fourier transforming this differential equation gives

$$m\mathbf{r}(\omega)\left(-\omega^2 - i\omega\Gamma + \omega_0^2\right) = -e\mathbf{E}(\omega)$$

Thus

$$\mathbf{r}(\omega) = \frac{e\mathbf{E}(\omega)}{m\left(\omega^2 - \omega_0^2 + i\omega\Gamma\right)}$$

The induced dipole moment per electron is **p** = -*e***r**, that is,

$$\mathbf{p}(\omega) = -\frac{e^2}{m}\frac{1}{\left(\omega^2 - \omega_0^2 + i\omega\Gamma\right)}\mathbf{E}(\omega)$$

Thus if we assume that there is one electron per atom and *n* atoms per unit volume we have for the atomic polarizability

$$\alpha^{at}(\omega) = -\frac{e^2}{m}\frac{1}{\left(\omega^2 - \omega_0^2 + i\omega\Gamma\right)},$$

for the polarizability

$$\alpha(\omega) = n\alpha^{at}(\omega)$$

and for the dielectric function

$$\begin{aligned}\varepsilon(\omega) &= 1 + 4\pi n\alpha^{at}(\omega)\\ &= 1 - \frac{4\pi ne^2}{m}\frac{1}{\left(\omega^2 - \omega_0^2 + i\omega\Gamma\right)}\\ &= 1 - \frac{\omega_{pl}^2}{\left(\omega^2 - \omega_0^2 + i\omega\Gamma\right)}\end{aligned}$$

where ω_{pl} is the plasma frequency.

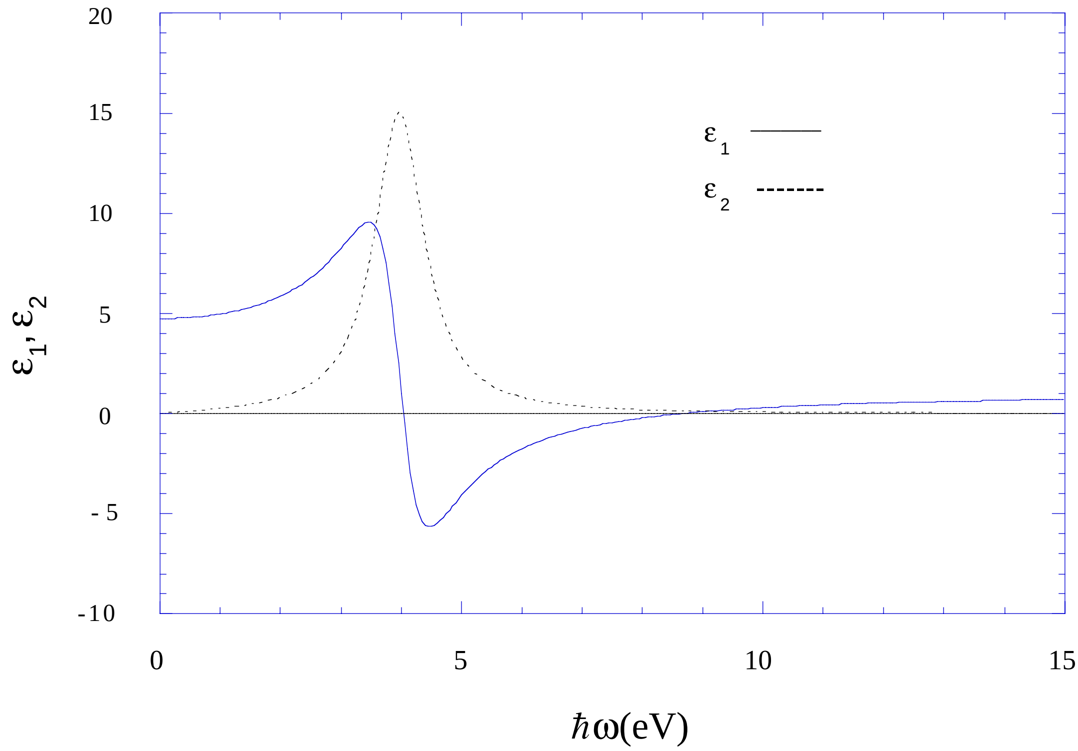
This treatment can be generalized to insulators with more than one electron per atom. We then have

$$\begin{aligned}\varepsilon(\omega) &= 1 - \frac{4\pi e^2}{m}\sum_j \frac{n_j}{\left(\omega^2 - \omega_j^2 + i\omega\Gamma_j\right)}\\ &= 1 - \sum_j \frac{\omega_{pl,j}^2}{\left(\omega^2 - \omega_j^2 + i\omega\Gamma_j\right)} \quad ; \quad \sum_j \omega_{pl,j}^2 = \omega_{pl}^2 = \frac{4\pi ne^2}{m}\end{aligned}$$

where n_j is the density of electrons with resonance frequency ω_j and n is the total density of electrons.

Let us study an example using this dielectric function with the values:

$$\begin{cases}\hbar\omega_0 = 4eV\\ \left(\hbar\omega_{pl}\right)^2 = 60(eV)^2\\ \hbar\Gamma = 1eV\end{cases}$$



Real and imaginary parts of a dielectric function in the Lorentz model.

Let us now study the *scattering of an electromagnetic wave by a bound electron*. We assume that the velocity of the electron will not be relativistic. Then the magnetic part of the Lorentz force can be neglected. The electric field will induce a dipole moment:

$$\mathbf{p}(\omega) = -\frac{e^2}{m}\frac{1}{\left(\omega^2 - \omega_0^2 + i\omega\Gamma\right)}\mathbf{E}(\omega)$$

The electron will be accelerated by the field and will radiate. Energy is absorbed from the incident wave by the electron and is then re-emitted into space. Such a process is called the *scattering* of the electromagnetic wave by the electron.

Now, from equations (9.15) and (9.16) we have for the power radiated per-unit-solid-angle by a time dependent dipole

$$\frac{dP}{d\Omega} = \frac{\left[\ddot{p}^2\right]}{4\pi c^3}\sin^2\theta$$

and for the total radiated power

$$P = \frac{2\left[\ddot{p}^2\right]}{3c^3}$$

Now using the rules for Fourier transforms we have

$$\ddot{\mathbf{p}}(\omega) = \frac{e^2}{m}\frac{\omega^2}{\left(\omega^2 - \omega_0^2 + i\omega\Gamma\right)}\mathbf{E}(\omega)$$

According to the *time-average product theorem* we have

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{1}{4\pi c^3}\frac{1}{2}\left[\frac{e^2}{m}\frac{\omega^2}{\left(\omega^2 - \omega_0^2 + i\omega\Gamma\right)}\right] \times \left[\frac{e^2}{m}\frac{\omega^2}{\left(\omega^2 - \omega_0^2 + i\omega\Gamma\right)}\right]^* E_0^2 \sin^2\theta$$

$$= \frac{1}{8\pi c^3}\left[\frac{\left(e^2\omega^2/m\right)^2}{\left(\omega^2 - \omega_0^2\right)^2 + (\omega\Gamma)^2}\right] E_0^2 \sin^2\theta$$

and

$$\langle P \rangle = \frac{1}{3c^3}\left[\frac{\left(e^2\omega^2/m\right)^2}{\left(\omega^2 - \omega_0^2\right)^2 + (\omega\Gamma)^2}\right] E_0^2$$

The scattering cross section, σ, is

$$\sigma \equiv \frac{\langle \text{reradiated power} \rangle}{\langle \text{incident power per area} \rangle}$$

It is the equivalent area of the incident wave front that delivers the power reradiated by the particle.

The denominator is the time average of the Poynting flux

$$S = \frac{c}{8\pi} E_0^2$$

Thus we have for the *differential scattering cross section*

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle = \frac{\left(e^2 \omega^2 / mc^2 \right)^2}{\left(\omega^2 - \omega_0^2 \right)^2 + (\omega\Gamma)^2} \sin^2 \theta$$

and for the total cross section

$$\langle \sigma \rangle = \frac{8\pi}{3} \frac{\left(e^2 \omega^2 / mc^2 \right)^2}{\left(\omega^2 - \omega_0^2 \right)^2 + (\omega\Gamma)^2}$$

The expression for the differential coss section is valid for linearly polarized light and θ is the angle between the dipole vector and the direction of the outgoing radiation.

The absorption lines are up in the *UV* range and in the visible range and below we have that $\omega << \omega_0$. This means that

$$\boxed{\sigma_{Rayleigh} = \frac{8\pi}{3}\left(\frac{e^2\omega^2}{mc^2\omega_0^2}\right)^2 = \frac{8\pi}{3}\left(\frac{4\pi^2 e^2}{m\omega_0^2}\right)\frac{1}{\lambda^4}}$$

Rayleigh scattering.

This explains why the sky is blue and the sunset red.

Now since the classical electron radius is

$$r_0 = \frac{e^2}{mc^2}$$

one expects classically to find the scattering cross section to be

$$\sigma_0 = \pi r_0^2 = \pi\left(\frac{e^2}{mc^2}\right)^2$$

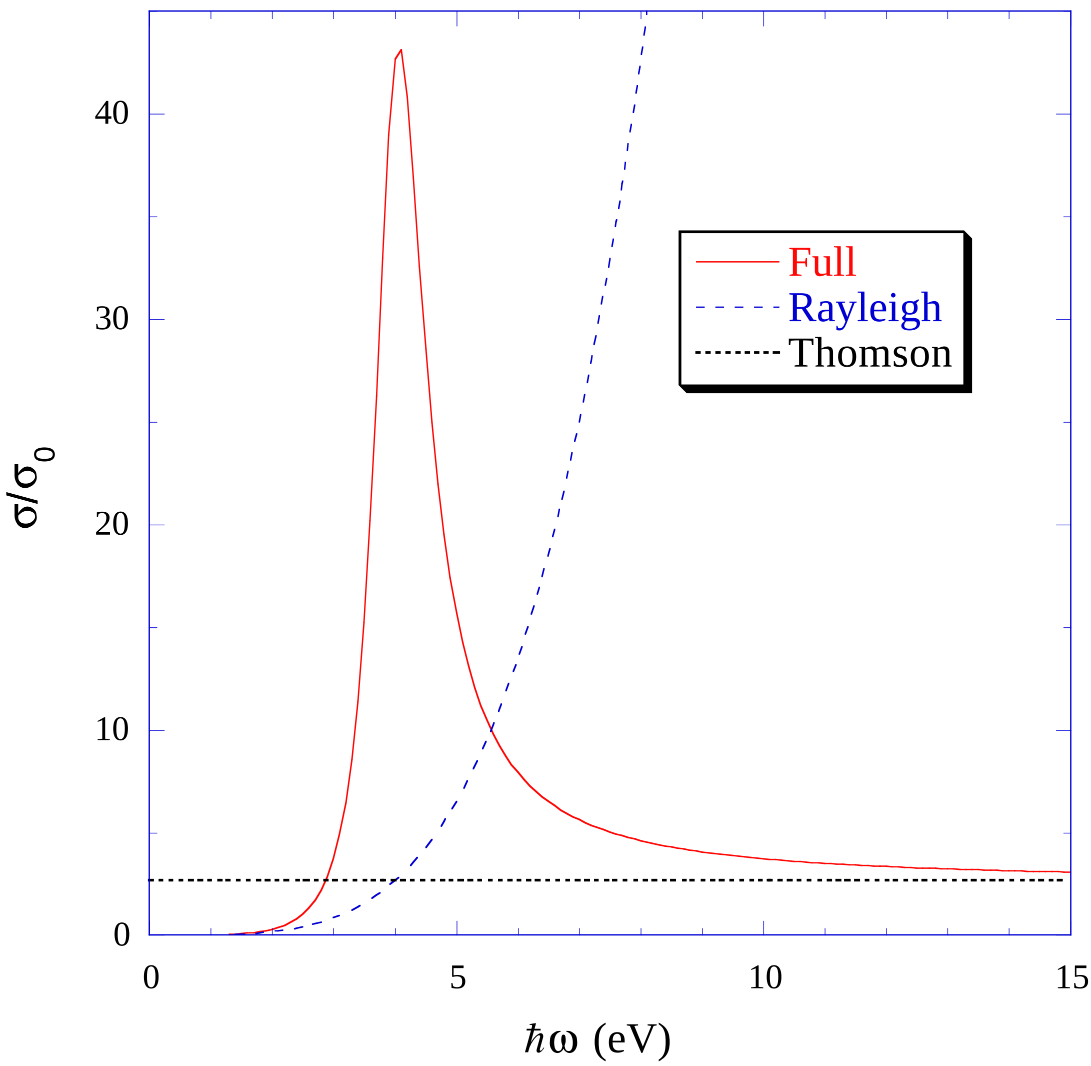
for a free, unbound, electron.

We find here that

$$\langle\sigma\rangle = \frac{8\sigma_0}{3}\frac{(\omega)^4}{\left(\omega^2 - \omega_0^2\right)^2 + (\omega\Gamma)^2}$$

and that

$$\sigma_{Rayleigh} = \frac{8\sigma_0}{3}\left(\frac{\omega}{\omega_0}\right)^4$$



The scattering cross section with the same parameters as before. We see that for small energies the Rayleigh scattering is valid. For high frequencies the scattering approaches that for a free, unbound, electron. Even higher up in energy, in the *x*-ray range, when the energy is of the order of the electron rest energy there are quantum effects modifying the result. The quantum mechanical result is known as the *Klein-Nishina formula*.

$$\sigma_{\text{K.N.}} = \left(\frac{e^2}{mc^2}\right)^2 \begin{cases} \dfrac{8\pi}{3}\left(1 - \dfrac{2\hbar\omega}{mc^2} + \cdots\right), & \hbar\omega << mc^2 \\ \pi \dfrac{mc^2}{\hbar\omega}\left[\ln\left(\dfrac{2\hbar\omega}{mc^2}\right) + \dfrac{1}{2}\right], & \hbar\omega >> mc^2 \end{cases}$$

Drude's classical model for the dielectric function of metals

The Drude model for metals is obtained from the Lorentz model by letting the electrons be free and not bound to the atoms. This is obtained by letting ω_0 be zero:

$$\varepsilon(\omega) = 1 - \frac{\omega_{pl}^2}{\left(\omega^2 + i\omega\Gamma\right)} = 1 - \frac{\omega_{pl}^2}{\omega(\omega + i\Gamma)}$$

This dielectric function works really well. In the generalized Drude model one lets Γ be frequency dependent. In doing so one has to let Γ be complex valued for ε to obey the Kramers Kronig dispersion relations. It turns out that the real part of Γ then stays rather constant all the way up to the plasma energy and the imaginary part is of less importance. Thus the simple Drude model is valid all the way up to the plasma energy.

Let us now study the *scattering of an electromagnetic wave by a free electron.*

We now have

$$\mathbf{p}(\omega) = -\frac{e^2}{m} \frac{1}{\omega(\omega + i\Gamma)} \mathbf{E}(\omega)$$

and

$$\ddot{\mathbf{p}}(\omega) = \frac{e^2}{m} \frac{\omega}{(\omega + i\Gamma)} \mathbf{E}(\omega)$$

If there are no frictional forces, no scattering against impurities or negligible radiation damping, the expressions are

$$\mathbf{p}(\omega) = -\frac{e^2}{m\omega^2} \mathbf{E}(\omega)$$

and

$$\ddot{\mathbf{p}}(\omega) = \frac{e^2}{m}\mathbf{E}(\omega)$$

Thus we have for the *differential scattering cross section*

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle = \left(\frac{e^2}{mc^2} \right)^2 \sin^2\theta = r_0^2 \sin^2\theta$$

and for the total cross section

$$\langle \sigma \rangle = \frac{8\pi}{3}\left(\frac{e^2}{mc^2} \right)^2 = \frac{8\pi}{3} r_0^2 = \frac{8}{3}\sigma_0$$

$$\boxed{\sigma_{\mathrm{Thomson}} = \frac{8\pi}{3}\left(\frac{e^2}{mc^2} \right)^2 \approx 0.665 \times 10^{-24}\, cm^2}$$

Thomson scattering.

Dielectric function for a metal

In a metallic system there are also bound charges contributing to the screening. The Drude result is then modified and we have:

$$\varepsilon(\omega) = 1 - \frac{\omega_{pl}^2}{\omega(\omega + i\Gamma)} \rightarrow$$

$$\tilde{\varepsilon}(\omega) = \varepsilon(\omega) - \frac{\omega_{pl}^2}{\omega\left[\omega + i(1/\tau)\right]} \quad ; \quad \tau \ \ \text{transport time}$$

The second term is related to the dynamical conductivity

$$\tilde{\varepsilon}(\omega) = \varepsilon(\omega) - \frac{4\pi n e^2}{m\omega\left[\omega + i(1/\tau)\right]} = \varepsilon(\omega) + \frac{4\pi i \sigma(\omega)}{\omega}$$

where

$$\sigma(\omega) = \frac{ne^2\tau}{m[1 - i\omega\tau]}$$

The static conductivity is

$$\sigma(0) = \frac{ne^2\tau}{m}$$

which you are familiar with.

Discussion: Here we have neglected the momentum dependence of the dielectric function. It is permissible in this course where we are concerned with optical properties or the interaction of electromagnetic waves with matter. The momentum of the photon is usually very small. We are concerned with processes very near the frequency axis of the ωq-plane.